Algorithms for Modules over an Exterior Algebra

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Introduction

- **Computational Algebra**
  - The terms **algebra** and **algorithm** originate from the Persian scientist al-Khorezmi [Grabmeier et al., 2003] who uses *al-gabr* and *muqabalah* to describe symbolic transformations and term reductions respectively, which are performed to solve algebraic equations.
  - Algorithmic manipulation of symbolic algebraic expressions remained to be the major task of algebra until the end of the 19th century. After that, this method was amended and shadowed by developments in abstract algebra. There, the main interest is focussed on formal investigations of algebraic structures derived from axioms.
  - During the past decades, algorithmic aspects of algebra became prevailing again. In particular, the accelerated development of computers and digital data processing made it feasible to automate manipulation of formulas and symbolic computations.

Introduction

- **Exterior Ideals**
  - We introduced a Macaulay2 [Grayson and Stillman, 2018] package that allows one to deal with classes of monomial ideals over an exterior algebra E in order to easily compute stable, strongly stable and lexsegment ideals.
  - Moreover, an algorithm to check whether an $(n+1)$-tuple $(1, h_1, \ldots, h_n)$ of nonnegative integers is the Hilbert function of a graded K–algebra of the form E/I, with I graded ideal of E, is given.
  - In particular, if $H_{E/I}$ is the Hilbert function of a graded K-algebra E/I, the package is able to construct the unique lexsegment ideal I^{lex} such that $H_{E/I} = H_{E/I^{\text{lex}}}$.
  - Finally, an algorithm to compute all the admissible Hilbert functions of graded K–algebras E/I, with given E, is also described.

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  - We are working on a Macaulay2 package that allows one to manage classes of monomial submodules of a finitely generated graded free module F over an exterior algebra E in order to compute stable, strongly stable and lexsegment modules.
  - We have implemented some algorithms to check whether a sequence of nonnegative integers is the Hilbert function of a graded E–module of the form F/M, with M graded submodule of F.
  - In particular, if $H_{F/M}$ is the Hilbert function of a graded E-module F/M, some appropriate methods are able to construct the unique lexsegment module M^{lex} such that $H_{F/M} = H_{F/M^{\text{lex}}}$.

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Exterior Algebra

- Let K be a field. We denote by $E = K\langle e_1, \ldots, e_n \rangle$ the exterior algebra of a K-vector space V with basis $e_1, \ldots, e_n$.
- For any subset $\sigma = \{i_1, \ldots, i_d\}$ of $\{1, \ldots, n\}$, with $i_1 < i_2 < \cdots < i_d$, we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \emptyset$.
The set of monomials in E forms a K-basis of E of cardinality 2^n.
- We put $fg = f \wedge g$ for any two elements f and g in E. An element $f \in E$ is called *homogeneous* of degree j if $f \in E_j$, where $E_j = \bigwedge^j V$.
- We define $\mathrm{supp}(e_\sigma) = \sigma = \{j : e_j \text{ divides } e_\sigma\}$ and $\mathrm{m}(e_\sigma) = \max\{i : i \in \mathrm{supp}(e_\sigma)\}$. Moreover, we set $\mathrm{m}(e_\sigma) = 0$ if $e_\sigma = 1$.

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Exterior Ideals

- If I is a graded ideal in E, then the function $H_I : \mathbb{N} \to \mathbb{N}$ given by $H_I(d) = \dim_K I_d$ $(i \geq 0)$ is called the Hilbert function of I.
- Let I be a monomial ideal of E. I is called stable if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$.
- I is called strongly stable if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.
- Let $>_{\mathrm{lex}}$ the *lexicographic order* on the set of all monomials of degree $d \geq 1$ in E. A monomial ideal I of E is called a **lexsegment ideal** (lex ideal, for short) if for all monomials $u \in I$ and all monomials $v \in E$ with $\deg u = \deg v$ and $v >_{\mathrm{lex}} u$, then $v \in I$.

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```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LL
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 :  loadPackage "ExteriorIdeals";

i2 :  E=QQ[e_1..e_5,SkewCommutative=>true];

i3 :  I=ideal {e_2*e_3,e_3*e_4*e_5}

o3 = ideal (e e , e e e )
             2 3   3 4 5

o3 :  Ideal of E

i4 :  Is=stableIdeal I

o4 = ideal (e e , e e e , e e , e e e )
             1 2   1 3 4   2 3   3 4 5

o4 :  Ideal of E

i5 :  Iss=stronglyStableIdeal Is

o5 = ideal (e e , e e , e e e , e e , e e e , e e e )
             1 2   1 3   1 4 5   2 3   2 4 5   3 4 5

o5 :  Ideal of E

i6 :  isLexIdeal Iss
o6 = false
```

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o3 :  Ideal of E
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i5 :  Iss=stronglyStableIdeal Is
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Exterior Ideals

- Let a and i be two positive integers. Then a has the unique i-th Macaulay expansion $a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$ with $a_i > a_{i-1} > \cdots a_j \geq j \geq 1$.
- We define $a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}$.
  We also set $0^{(i)} = 0$ for all $i \geq 1$.

Thm Let $(h_1, \ldots, h_n)$ be a sequence of nonnegative integers. Then the following conditions are equivalent:

(a) $1 + \sum_{i=1}^{n} h_i t^i$ is the Hilbert series of a graded K–algebra E/I;

(b) $0 < h_{i+1} \leq h_i^{(i)}$, $0 < i \leq n-1$.

This theorem is known as the Kruskal–Katona theorem [Aramova et al., 1997].

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Exterior Ideals

- If $(1, h_1, \ldots, h_n)$ is a sequence of nonnegative integers such that
  (i) $h_1 \leq n$,
  (ii) $0 < h_{i+1} \leq h_i^{(i)}$, $0 < i \leq n-1$,

  then there exists a unique lex ideal I of an exterior algebra E with n generators over a field K such that $H_{E/I}(d) = h_d$ $(d = 0, \ldots, n)$.
- The sequence $(1, h_1, \ldots, h_n)$ is called the Hilbert sequence [Amata and Crupi, 2018b] of E/I. We will denote it by $Hs_{E/I}$.
  - Note that if $I = 0$, then $Hs_{E/I} = Hs_E = (1, n, \binom{n}{2}, \cdots, \binom{n}{n})$.
  - Furthermore, we set $Hs_{E/I} = (\underbrace{0, \ldots, 0}_{n+1})$, if $I = E$.

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i1 :  E=QQ[e_1..e_5,SkewCommutative=>true];
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o2 :  false
i3 :  lexIdeal({1,6,3,0,0,0,0},E)
stdio:24:1:(3):  error:  expected a Hilbert sequence
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o5 :  Ideal of E
i6 :  hilbertSequence I
```

o6 = {1, 5, 7, 4, 0, 0}

```
o6 :  List
i7 :  lexIdeal({1,5,7,4,0,0},E)
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o7 = ideal $(e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4e_5)$

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i1 :  E=QQ[e_1..e_4,SkewCommutative=>true];

i2 :  hilbSeqs=allHilbertSequences(E)

o2 :  {{1,4,6,4,1}, {1,4,6,4,0}, {1,4,6,3,0}, {1,4,6,2,0}, {1,4,6,1,0},
       -----------------------------------------------------------------
       {1,4,6,0,0}, {1,4,5,2,0}, {1,4,5,1,0}, {1,4,5,0,0}, {1,4,4,1,0},
       -----------------------------------------------------------------
       {1,4,4,0,0}, {1,4,3,1,0}, {1,4,3,0,0}, {1,4,2,0,0}, {1,4,1,0,0},
       -----------------------------------------------------------------
       {1,4,0,0,0}, {1,3,3,1,0}, {1,3,3,0,0}, {1,3,2,0,0}, {1,3,1,0,0},
       -----------------------------------------------------------------
       {1,3,0,0,0}, {1,2,1,0,0}, {1,2,0,0,0}, {1,1,0,0,0}, {1,0,0,0,0}}
o2 :  List

i3 :  transpose matrix hilbSeqs

o3 = | 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 |
     | 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 3 3 3 3 3 2 2 1 0 |
     | 6 6 6 6 6 6 5 5 5 4 4 3 3 2 1 0 3 3 2 1 0 1 0 0 0 |
     | 4 4 3 2 1 0 2 1 0 1 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 |
     | 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |

             5       25
o3 :  Matrix ZZ <--- ZZ
```

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Exterior Modules

- Let $\mathcal{M}$ be the category of finitely generated $\mathbb{Z}$-graded left and right E-modules M. For all $M \in \mathcal{M}$, the function $H_M : \mathbb{Z} \to \mathbb{Z}$ given by $H_M(d) = \dim_K M_d$ is called the Hilbert function of M.
- Let $F \in \mathcal{M}$ be a free module with homogeneous basis $g_1, \dots, g_r$, where $\deg(g_i) = f_i$ for each $i = 1, \dots, r$, with $f_1 \leq f_2 \leq \cdots \leq f_r$. We write $F = \oplus_{i=1}^r E g_i$.
- The elements of the form $e_\sigma g_i$, where $e_\sigma \in \mathrm{Mon}(E)$, are called *monomials* of F, and $\deg(e_\sigma g_i) = \deg(e_\sigma) + \deg(g_i)$.
- In particular, if $F = E^r$ and $g_i = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 appears in the i-th place, we assume, as usual, $\deg(e_\sigma g_i) = \deg(e_\sigma)$, *i.e.*, $\deg(g_i) = f_i = 0$, for all i.

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Exterior Modules

- Let $\mathcal{M}$ be the category of finitely generated $\mathbb{Z}$-graded left and right E-modules M. For all $M \in \mathcal{M}$, the function $H_M : \mathbb{Z} \to \mathbb{Z}$ given by $H_M(d) = \dim_K M_d$ is called the Hilbert function of M.
- Let $F \in \mathcal{M}$ be a free module with homogeneous basis $g_1, \ldots, g_r$, where $\deg(g_i) = f_i$ for each $i = 1, \ldots, r$, with $f_1 \leq f_2 \leq \cdots \leq f_r$. We write $F = \oplus_{i=1}^{r} E g_i$.
- The elements of the form $e_\sigma g_i$, where $e_\sigma \in \mathrm{Mon}(E)$, are called *monomials* of F, and $\deg(e_\sigma g_i) = \deg(e_\sigma) + \deg(g_i)$.
- In particular, if $F = E^r$ and $g_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 appears in the i-th place, we assume, as usual, $\deg(e_\sigma g_i) = \deg(e_\sigma)$, *i.e.*, $\deg(g_i) = f_i = 0$, for all i.

Exterior Modules

- A graded submodule M of F is a **monomial** submodule if M is a submodule generated by monomials of F: $M = I_1 g_1 \oplus \cdots \oplus I_r g_r$, with I_i a monomial ideal of E, for each i.
- A monomial submodule $M = \oplus_{i=1}^{r} I_i g_i$ of F is a **(strongly) stable** submodule if I_i is a (strongly) stable ideal of E, for each i, and $(e_1, \ldots, e_n)^{f_{i+1}-f_i} I_{i+1} \subseteq I_i$, for $i = 1, \ldots, r-1$.
- Let $>_{\mathrm{lex}_F}$ the POT extension in F of the *lexicographic order* $>_{\mathrm{lex}}$ in E. Let $\mathcal{L}$ be a monomial submodule of F. $\mathcal{L}$ is a **lexicographic submodule** (lex submodule, for short) if for all $u, v \in \mathrm{Mon}_d(F)$ with $u \in \mathcal{L}$ and $v >_{\mathrm{lex}_F} u$, one has $v \in \mathcal{L}$, for every $d \geq 1$.

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```
i1 :  loadPackage "ExteriorModules";

i2 :  E=QQ[e_1..e_5,SkewCommutative=>true];

i3 :  F=E^2;

i4 :  I_1=ideal {e_1*e_2, e_1*e_3, e_1*e_4*e_5};

i5 :  I_2=ideal {e_1*e_2, e_2*e_3*e_4};

i6 :  M=createModule({I_1, I_2},F)

o6 = image | e_1e_3 e_1e_2 e_1e_4e_5 0      0        |
           | 0      0      0         e_1e_2 e_2e_3e_4 |

                               2
o6 :  E-module, submodule of E

i7 :  Ms=stableModule M

o7 = image | e_1e_2 e_1e_3 e_1e_4e_5 e_2e_3e_4 0      0        |
           | 0      0      0         0         e_1e_2 e_2e_3e_4 |

                               2
o7 :  E-module, submodule of E

i8 :  Mss=stronglyStableModule M

o8 = image | e_1e_2 e_1e_3 e_1e_4e_5 e_2e_3e_4 0      0         0        |
           | 0      0      0         0         e_1e_2 e_1e_3e_4 e_2e_3e_4 |

                               2
o8 :  E-module, submodule of E
```

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```
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i6 :  M=createModule({I_1, I_2},F)
o6 = image | e_1e_3  e_1e_2  e_1e_4e_5  0       0         |
           | 0       0       0          e_1e_2  e_2e_3e_4 |
                                     2
o6 :  E-module, submodule of E
i7 :  Ms=stableModule M
o7 = image | e_1e_2  e_1e_3  e_1e_4e_5  e_2e_3e_4  0       0         |
           | 0       0       0          0          e_1e_2  e_2e_3e_4 |
                                     2
o7 :  E-module, submodule of E
i8 :  Mss=stronglyStableModule M
o8 = image | e_1e_2  e_1e_3  e_1e_4e_5  e_2e_3e_4  0       0           0         |
           | 0       0       0          0          e_1e_2  e_1e_3e_4   e_2e_3e_4 |
                                     2
o8 :  E-module, submodule of E
```

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i6 :  M=createModule({I_1, I_2},F)
o6 = image | e_1e_3 e_1e_2 e_1e_4e_5 0      0        |
           | 0      0      0         e_1e_2 e_2e_3e_4 |
                                   2
o6 :  E-module, submodule of E
i7 :  Ms=stableModule M
o7 = image | e_1e_2 e_1e_3 e_1e_4e_5 e_2e_3e_4 0      0        |
           | 0      0      0         0         e_1e_2 e_2e_3e_4 |
                                   2
o7 :  E-module, submodule of E
i8 :  Mss=stronglyStableModule M
o8 = image | e_1e_2 e_1e_3 e_1e_4e_5 e_2e_3e_4 0      0         0        |
           | 0      0      0         0         e_1e_2 e_1e_3e_4 e_2e_3e_4 |
                                   2
o8 :  E-module, submodule of E
```

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Exterior Modules

- If I is a graded ideal of E, then $I = \oplus_{j \geq 0} I_j$, where I_j is the K-vector space of all homogeneous elements of degree j.
  We denote by $\text{indeg}\, I$ the initial degree of I, that is, the minimum s such that $I_s \neq 0$.

- Let $\mathcal{L}$ be a graded submodule of F. Then $\mathcal{L}$ is a lex submodule of F if and only if
  (i) $\mathcal{L} = \oplus_{i=1}^{r} I_i g_i$, with I_i lex ideals of E , for $i = 1, \ldots, r$, and
  (ii) $(e_1, \ldots, e_n)^{\rho_i + f_i - f_{i-1}} \subseteq I_{i-1}$, for $i = 2, \ldots, r$, with $\rho_i = \text{indeg}\, I_i$.

Thm Let $(h_1, \ldots, h_n)$ be a sequence of non negative integers. Then the following conditions are equivalent [Amata and Crupi, 2018a]:

(a) $r + \sum_{i=1}^{n} h_i t^i$ is the Hilbert series of a graded E-module E^r/M, $r \geq 1$;

(b) $h_i = \sum_{j=1}^{r} h_{i,j}$, for $i = 1, \ldots, n$, and $(h_{1,j}, h_{2,j}, \ldots, h_{n,j})$ is an n-tuple of non negative integers such that $0 < h_{i+1,j} \leq h_{i,j}^{(i)}$, for $0 < i \leq n-1$ and $j = 1, \ldots, r$, $r \geq 1$;

(c) there exists a unique lexicographic submodule $\mathcal{L}$ of F such that $r + \sum_{i=1}^{n} h_i t^i$ is the Hilbert series of $E^r/\mathcal{L}$, $r \geq 1$.

Exterior Modules

- If I is a graded ideal of E, then $I = \oplus_{j\geq 0} I_j$, where I_j is the K-vector space of all homogeneous elements of degree j.
  We denote by $\operatorname{indeg} I$ the initial degree of I, that is, the minimum s such that $I_s \neq 0$.

- Let $\mathcal{L}$ be a graded submodule of F. Then $\mathcal{L}$ is a lex submodule of F if and only if
  (i) $\mathcal{L} = \oplus_{i=1}^{r} I_i g_i$, with I_i lex ideals of E , for $i = 1, \ldots, r$, and
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Thm Let $(h_1, \ldots, h_n)$ be a sequence of non negative integers. Then the following conditions are equivalent [Amata and Crupi, 2018a]:

(a) $r + \sum_{i=1}^{n} h_i t^i$ is the Hilbert series of a graded E-module E^r/M, $r \geq 1$;

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or

```
i1 :  loadPackage "ExteriorModules";
i2 :  E=QQ[e_1..e_4,SkewCommutative=>true];
i3 :  F=E^3;
i4 :  I_1=ideal {e_1, e_2*e_3*e_4};
i5 :  I_2=ideal {e_1*e_2, e_1*e_3*e_4};
i6 :  I_3=ideal {e_1*e_2*e_3};
i7 :  M=createModule({I_1, I_2, I_3},F)
```

$$o7 = \text{image} \left| \begin{array}{lllll} e_1 & e_2e_3e_4 & 0 & 0 & 0 \\ 0 & 0 & e_1e_2 & e_1e_3e_4 & 0 \\ 0 & 0 & 0 & 0 & e_1e_2e_3 \end{array} \right|$$

3

```
o7 :  E-module, submodule of E
i8 :  L=lexModule M
```

$$o8 = \text{image} \left| \begin{array}{lllllll} e_1 & e_2e_3 & 0 & 0 & 0 & & \\ 0 & 0 & e_1e_2e_3 & e_1e_2e_4 & e_1e_3e_4 & e_2e_3e_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_1e_2e_3e_4 \end{array} \right|$$

3

```
o8 :  E-module, submodule of E
i9 :  hilbertSequence M
```

$o9 = \{3, 11, 14, 4, 0\}$

```
i10 :  hilbertSequence M==hilbertSequence L
o10 = true
```

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or

```
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i2 :  E=QQ[e_1..e_4,SkewCommutative=>true];

i3 :  F=E^3;

i4 :  I_1=ideal {e_1, e_2*e_3*e_4};

i5 :  I_2=ideal {e_1*e_2, e_1*e_3*e_4};

i6 :  I_3=ideal {e_1*e_2*e_3};

i7 :  M=createModule({I_1, I_2, I_3},F)

o7 = image | e_1  e_2e_3e_4 0      0          0          |
           | 0    0         e_1e_2 e_1e_3e_4  0          |
           | 0    0         0      0          e_1e_2e_3  |
                                  3
o7 :  E-module, submodule of E

i8 :  L=lexModule M

o8 = image | e_1  e_2e_3  0          0          0          0          0             |
           | 0    0       e_1e_2e_3  e_1e_2e_4  e_1e_3e_4  e_2e_3e_4  0             |
           | 0    0       0          0          0          0          e_1e_2e_3e_4  |
                                  3
o8 :  E-module, submodule of E

i9 :  hilbertSequence M

o9 = {3, 11, 14, 4, 0}

i10 :  hilbertSequence M==hilbertSequence L
o10 = true
```

Exterior Modules

- Let F_d be the part of degree d of $F = \oplus_{i=1}^{r} Eg_i$ and denote by $\mathrm{Mon}_d(F)$ the set of all monomials of degree d of F.
- Let $E = K\langle e_1, e_2, e_3\rangle$ and $F = Eg_1 \oplus Eg_2$, with $\deg g_1 = 2$ and $\deg g_2 = 3$, the monomials of F, with respect to $>_{\mathrm{lex}_F}$, are ordered as follows:

$\mathrm{Mon}_2(F)$	g_1
$\mathrm{Mon}_3(F)$	$e_1g_1 >_{\mathrm{lex}_F} e_2g_1 >_{\mathrm{lex}_F} e_3g_1 >_{\mathrm{lex}_F} g_2$
$\mathrm{Mon}_4(F)$	$e_1e_2g_1 >_{\mathrm{lex}_F} e_1e_3g_1 >_{\mathrm{lex}_F} e_2e_3g_1 >_{\mathrm{lex}_F} e_1g_2 >_{\mathrm{lex}_F} e_2g_2 >_{\mathrm{lex}_F} e_3g_2$
$\mathrm{Mon}_5(F)$	$e_1e_2e_3g_1 >_{\mathrm{lex}_F} e_1e_2g_2 >_{\mathrm{lex}_F} e_1e_3g_2 >_{\mathrm{lex}_F} e_2e_3g_2$
$\mathrm{Mon}_6(F)$	$e_1e_2e_3g_2$

Exterior Modules

- Assume M is a monomial submodule of $F = \oplus_{i=1}^{r} Eg_i$.
  One can quickly verify that $H_F(d) = \dim_K F_d = 0$, for $d < f_1$ and $d > f_r + n$. Hence, it follows that

$$H_{F/M}(t) = \sum_{i=f_1}^{f_r+n} H_{F/M}(i)t^i,$$

  and we can associate to F/M the following sequence

$$(H_{F/M}(f_1), H_{F/M}(f_1+1), \ldots, H_{F/M}(f_r+n)) \in \mathbb{N}_0^{f_r+n-f_1+1}.$$

- Such a sequence is called the Hilbert sequence of F/M, and denoted by $Hs_{F/M}$ [Amata and Crupi, 2018c].
  The integers $f_1, f_1+1, \ldots, f_r+n$ are called the $Hs_{F/M}$-degrees.

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  The integers $f_1, f_1+1, \ldots, f_r+n$ are called the $Hs_{F/M}$-degrees.

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```
i1 :  loadPackage "ExteriorModules";
i2 :  E=QQ[e_1..e_4,SkewCommutative=>true];
i3 :  F=E^{2,1,-3};
i4 :  I_1=ideal {e_1*e_2, e_1*e_3, e_2*e_3*e_4};
i5 :  I_2=ideal {e_1*e_2, e_1*e_3, e_1*e_4, e_2*e_3};
i6 :  I_3=ideal {e_1*e_2*e_3, e_1*e_2*e_4};
i7 :  M=createModule({I_1, I_2, I_3},F)
o7 = image | e_1e_2 e_1e_3 e_2e_3e_4 0      0      0      0      0          0          |
           | 0      0      0         e_1e_2 e_1e_3 e_1e_4 e_2e_3 0          0          |
           | 0      0      0         0      0      0      0      e_1e_2e_3  e_1e_2e_4  |
                                3
o7 :  E-module, submodule of E
i8 :  isLexModule M
o8 = true
i9 :  hilbertSequence M
o9 = {1, 5, 8, 2, 0, 1, 4, 6, 2, 0}
o9 :  List
```

RESET INTERRUPT SAVE UPLOAD FILE

```
i1 :  loadPackage "ExteriorModules";
i2 :  E=QQ[e_1..e_4,SkewCommutative=>true];
i3 :  F=E^{2,1,-3};
i4 :  I_1=ideal {e_1*e_2, e_1*e_3, e_2*e_3*e_4};
i5 :  I_2=ideal {e_1*e_2, e_1*e_3, e_1*e_4, e_2*e_3};
i6 :  I_3=ideal {e_1*e_2*e_3, e_1*e_2*e_4};
i7 :  M=createModule({I_1, I_2, I_3},F)
o7 = image | e_1e_2 e_1e_3 e_2e_3e_4 0      0      0      0      0        0        |
           | 0      0      0         e_1e_2 e_1e_3 e_1e_4 e_2e_3 0        0        |
           | 0      0      0         0      0      0      0      e_1e_2e_3 e_1e_2e_4 |
                                  3
o7 :  E-module, submodule of E
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o9 = {1, 5, 8, 2, 0, 1, 4, 6, 2, 0}
o9 :  List
```

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Exterior Modules

- Let M be a graded submodule of $F = \oplus_{i=1}^{r} Eg_i$ and let $H_{F/M}$ the Hilbert function of F/M. There exists an integer $N \leq r$ such that we have the unique expression

$$H_{F/M}(d) = \sum_{i=N+1}^{r} \binom{n}{d - f_i} + a,$$

where

$$a = \binom{a_0}{d - f_N} + \binom{a_1}{d - f_N - 1} + \cdots + \binom{a_s}{d - f_N - s} < \binom{n}{d - f_N}$$

is the Macaulay representation of a lex_F segment in degree $d - f_N$ in the N-th component of F.

- Moreover,

$$H_{F/M}(d+1) \leq \sum_{i=N+1}^{r} \binom{n}{d - f_i + 1} + a^{(d - f_N)},$$

for $d \geq \mathrm{indeg} Hs_{F/M} + 1$.

Exterior Modules

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where

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Exterior Modules

Thm Let $(f_1, f_2, \ldots, f_r) \in \mathbb{Z}^r$ be an r–tuple such that $f_1 \leq f_2 \leq \cdots \leq f_r$ and let $(h_{f_1}, h_{f_1+1}, \ldots, h_{f_r+n})$ be a sequence of nonnegative integers.

Set $s = \min\{k \in [f_1, f_r + n] : h_k \neq 0\}$,

and $\tilde{r}_j = |\{p \in [r] : f_p = s + j\}|, \quad \text{for} \quad j = 0, 1.$

Then the following conditions are equivalent:

(a) $\sum_{i=s}^{f_r+n} h_i t^i$ is the Hilbert series of a graded E-module F/M, with $F = \oplus_{i=1}^r Eg_i$ finitely generated graded free E-module with the basis elements g_i of degrees f_i;

(b) $h_s \leq \tilde{r}_0$, $h_{s+1} \leq n\tilde{r}_0 + \tilde{r}_1$, $h_i = \sum_{j=N+1}^r \binom{n}{i-f_j} + a$, where a is a positive integer less than $\binom{n}{i-f_N}$, $0 < N \leq r$, and $h_{i+1} \leq \sum_{j=N+1}^r \binom{n}{i-f_j+1} + a^{(i-f_N)}$, $i = s+1, \ldots, f_r + n$;

(c) there exists a unique lexicographic submodule L of a finitely generated graded free E-module $F = \oplus_{i=1}^r Eg_i$ with the basis elements g_i of degrees f_i and such that $\sum_{i=s}^{f_r+n} h_i t^i$ is the Hilbert series of F/L.

Exterior Modules

Thm Let $(f_1, f_2, \ldots, f_r) \in \mathbb{Z}^r$ be an r–tuple such that $f_1 \le f_2 \le \cdots \le f_r$ and let $(h_{f_1}, h_{f_1+1}, \ldots, h_{f_r+n})$ be a sequence of nonnegative integers.

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(c) there exists a unique lexicographic submodule L of a finitely generated graded free E-module $F = \oplus_{i=1}^r Eg_i$ with the basis elements g_i of degrees f_i and such that $\sum_{i=s}^{f_r+n} h_i t^i$ is the Hilbert series of F/L.

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```
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i7 :  M=createModule({I_1, I_2, I_3},F)
o7 = image | e_1e_2 e_3e_4 0      0          0          |
           | 0      0      e_1e_2 e_2e_3e_4  0          |
           | 0      0      0      0          e_2e_3e_4  |
                                3
o7 :  E-module, submodule of E
i8 :  L=lexModule M
o8 = image | e_1e_2 e_1e_3 e_1e_4 e_2e_3e_4 0         0         0         0         0            |
           | 0      0      0      0         e_1e_2e_3 e_1e_2e_4 e_1e_3e_4 e_2e_3e_4 0            |
           | 0      0      0      0         0         0         0         0         e_1e_2e_3e_4 |
                                3
o8 :  E-module, submodule of E
i9 :  hilbertSequence M
o9 = {3, 12, 15, 4, 0}
o9 :  List
```

RESET INTERRUPT SAVE UPLOAD FILE

```
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i4 :  I_1=ideal {e_1*e_2, e_3*e_4};
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i6 :  I_3=ideal {e_2*e_3*e_4};
i7 :  M=createModule({I_1, I_2, I_3},F)
```

$$o7 = \text{image} \begin{vmatrix} e\_1e\_2 & e\_3e\_4 & 0 & 0 & 0 \\ 0 & 0 & e\_1e\_2 & e\_2e\_3e\_4 & 0 \\ 0 & 0 & 0 & 0 & e\_2e\_3e\_4 \end{vmatrix}$$

```
                        3
o7 :  E-module, submodule of E
i8 :  L=lexModule M
```

$$o8 = \text{image} \begin{vmatrix} e\_1e\_2 & e\_1e\_3 & e\_1e\_4 & e\_2e\_3e\_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e\_1e\_2e\_3 & e\_1e\_2e\_4 & e\_1e\_3e\_4 & e\_2e\_3e\_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e\_1e\_2e\_3e\_4 \end{vmatrix}$$

```
                        3
o8 :  E-module, submodule of E
i9 :  hilbertSequence M
```

$o9 = \{3, 12, 15, 4, 0\}$

```
09 :  List
```

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RESET INTERRUPT SAVE UPLOAD FILE

```
i1 :  loadPackage "ExteriorModules";
i2 :  E=QQ[e_1..e_4,SkewCommutative=>true];
i3 :  F=E^{2,0,-2};
i4 :  I_1=ideal {e_1*e_2, e_3*e_4};
i5 :  I_2=ideal {e_1*e_2, e_2*e_3*e_4};
i6 :  I_3=ideal {e_2*e_3*e_4};
i7 :  M=createModule({I_1, I_2, I_3},F)
o7 = image | e_1e_2 e_3e_4 0      0        0        |
           | 0      0      e_1e_2 e_2e_3e_4 0        |
           | 0      0      0      0        e_2e_3e_4 |
                                 3
o7 :  E-module, submodule of E
i8 :  L=lexModule M
o8 = image | e_1e_2 e_1e_3 e_2e_3e_4 0      0        0        |
           | 0      0      0        e_1e_2 e_1e_3e_4 0        |
           | 0      0      0        0      0        e_1e_2e_3 |
                                 3
o8 :  E-module, submodule of E
i9 :  hilbertSequence M
o9 = {1, 4, 5, 4, 6, 5, 6, 3, 0}
o9 :  List
```

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RESET INTERRUPT SAVE UPLOAD FILE

```
i1 :  loadPackage "ExteriorModules";
i2 :  E=QQ[e_1..e_4,SkewCommutative=>true];
i3 :  F=E^{2,0,-2};
i4 :  I_1=ideal {e_1*e_2, e_3*e_4};
i5 :  I_2=ideal {e_1*e_2, e_2*e_3*e_4};
i6 :  I_3=ideal {e_2*e_3*e_4};
i7 :  M=createModule({I_1, I_2, I_3},F)
```

$$o7 = \text{image} \begin{vmatrix} e\_1e\_2 & e\_3e\_4 & 0 & 0 & 0 \\ 0 & 0 & e\_1e\_2 & e\_2e\_3e\_4 & 0 \\ 0 & 0 & 0 & 0 & e\_2e\_3e\_4 \end{vmatrix}$$

```
                           3
o7 :  E-module, submodule of E
i8 :  L=lexModule M
```

$$o8 = \text{image} \begin{vmatrix} e\_1e\_2 & e\_1e\_3 & e\_2e\_3e\_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & e\_1e\_2 & e\_1e\_3e\_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & e\_1e\_2e\_3 \end{vmatrix}$$

```
                           3
o8 :  E-module, submodule of E
i9 :  hilbertSequence M
```

$o9 = \{1, 4, 5, 4, 6, 5, 6, 3, 0\}$

```
09 :  List
```

RESET INTERRUPT SAVE UPLOAD FILE

```
i1 :  loadPackage "ExteriorModules";
i2 :  E=QQ[e_1..e_4,SkewCommutative=>true];
i3 :  F=E^3;
i4 :  hs={3, 12, 15, 4, 0};
i5 :  lexModule(hs,F)
o5 = image | e_1e_2 e_1e_3 e_1e_4 e_2e_3e_4 0        0        0        0        0          |
           | 0      0      0      0         e_1e_2e_3 e_1e_2e_4 e_1e_3e_4 e_2e_3e_4 0     |
           | 0      0      0      0         0        0        0        0        e_1e_2e_3e_4 |
                                  3
o5 :  E-module, submodule of E
i6 :  F=E^{2,0,-2};
i7 :  hs={1, 4, 5, 4, 6, 5, 6, 3, 0};
i8 :  lexModuleBySequences(hs,F)
o8 = image | e_1e_2 e_1e_3 e_2e_3e_4 0      0         0         |
           | 0      0      0         e_1e_2 e_1e_3e_4 0         |
           | 0      0      0         0      0         e_1e_2e_3 |
                                  3
o8 :  E-module, submodule of E
i9 :  F=E^{3,1,-2};
i10 :  hs={1, 2, 2, 4, 3, 3, 4, 5, 2, 0};
i11 :  isHilbertSequence(hs,F)
o11 = false
```

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RESET INTERRUPT SAVE UPLOAD FILE

```
i1 :  loadPackage "ExteriorModules";
i2 :  E=QQ[e_1..e_4,SkewCommutative=>true];
i3 :  F=E^3;
i4 :  hs={3, 12, 15, 4, 0};
i5 :  lexModule(hs,F)
o5 = image | e_1e_2 e_1e_3 e_1e_4 e_2e_3e_4 0        0        0        0        0          |
           | 0      0      0      0         e_1e_2e_3 e_1e_2e_4 e_1e_3e_4 e_2e_3e_4 0          |
           | 0      0      0      0         0        0        0        0        e_1e_2e_3e_4 |
                                3
o5 :  E-module, submodule of E
i6 :  F=E^{2,0,-2};
i7 :  hs={1, 4, 5, 4, 6, 5, 6, 3, 0};
i8 :  lexModuleBySequences(hs,F)
o8 = image | e_1e_2 e_1e_3 e_2e_3e_4 0      0         0         |
           | 0      0      0         e_1e_2 e_1e_3e_4 0         |
           | 0      0      0         0      0         e_1e_2e_3 |
                                3
o8 :  E-module, submodule of E
i9 :  F=E^{3,1,-2};
i10 :  hs={1, 2, 2, 4, 3, 3, 4, 5, 2, 0};
i11 :  isHilbertSequence(hs,F)
o11 = false
```

RESET INTERRUPT SAVE UPLOAD FILE

```
i1 :  loadPackage "ExteriorModules";
i2 :  E=QQ[e_1..e_4,SkewCommutative=>true];
i3 :  F=E^3;
i4 :  hs={3, 12, 15, 4, 0};
i5 :  lexModule(hs,F)
o5 = image | e_1e_2 e_1e_3 e_1e_4 e_2e_3e_4 0        0        0        0        0          |
           | 0      0      0      0        e_1e_2e_3 e_1e_2e_4 e_1e_3e_4 e_2e_3e_4 0          |
           | 0      0      0      0        0        0        0        0        e_1e_2e_3e_4 |
                              3
o5 :  E-module, submodule of E
i6 :  F=E^{2,0,-2};
i7 :  hs={1, 4, 5, 4, 6, 5, 6, 3, 0};
i8 :  lexModuleBySequences(hs,F)
o8 = image | e_1e_2 e_1e_3 e_2e_3e_4 0      0        0        |
           | 0      0      0        e_1e_2 e_1e_3e_4 0        |
           | 0      0      0        0      0        e_1e_2e_3 |
                              3
o8 :  E-module, submodule of E
i9 :  F=E^{3,1,-2};
i10 :  hs={1, 2, 2, 4, 3, 3, 4, 5, 2, 0};
i11 :  isHilbertSequence(hs,F)
o11 = false
```

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Lex–Algorithm

- Let $E = K\langle e_1, e_2, e_3, e_4\rangle$, $F = E^3$ and let us consider the sequence $H = (3, 12, 15, 4, 0)$.

H–degrees	**0**	1	2	3	4	
H	(3,	12,	15,	4,	0)	$-$
Hs_{E/I_3}	**(1,**	**4,**	**6,**	**4,**	**0)**	$-$
Hs_{E/I_2}	**(1,**	**4,**	**6,**	**0,**	**0)**	$-$
Hs_{E/I_1}	**(1,**	**4,**	**3,**	**0,**	**0)**	$=$
0_5	(0,	0,	0,	0,	0)	

- $M^{\text{lex}} = \oplus_i^r I_i g_i$ is the unique lex submodule with Hilbert sequence H.

$$M^{\text{lex}} = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2 \oplus (e_1e_2e_3e_4)g_3.$$

Lex–Algorithm

- Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = \oplus_{i=1}^{3} E g_i$ with $f_1 = -2, f_2 = 0, f_3 = 2$. Let us consider the $[-2, 6]$–sequence $H = (1, 4, 5, 4, 6, 5, 6, 3, 0)$.

H–degrees	$-\mathbf{2}$	-1	$\mathbf{0}$	1	$\mathbf{2}$	3	4	5	6	
H	$(1,$	$4,$	$5,$	$4,$	$6,$	$5,$	$6,$	$3,$	$0)$	$-$
$\widetilde{H}_3$	$(0,$	$0,$	$0,$	$0,$	$\mathbf{1},$	$\mathbf{4},$	$\mathbf{6},$	$\mathbf{3},$	$\mathbf{0})$	$-$
$\widetilde{H}_2$	$(0,$	$0,$	$\mathbf{1},$	$\mathbf{4},$	$\mathbf{5},$	$\mathbf{1},$	$\mathbf{0},$	$0,$	$0)$	$-$
$\widetilde{H}_1$	$(\mathbf{1},$	$\mathbf{4},$	$\mathbf{4},$	$\mathbf{0},$	$\mathbf{0},$	$0,$	$0,$	$0,$	$0)$	$=$
0_9	$(0,$	$0,$	$0,$	$0,$	$0,$	$0,$	$0,$	$0,$	$0)$	

- $M^{\text{lex}} = \oplus_i^r I_i g_i$ is the unique lex submodule with Hilbert sequence H.

$$M^{\text{lex}} = (e_1 e_2, e_1 e_3, e_2 e_3 e_4) g_1 \oplus (e_1 e_2, e_1 e_3 e_4) g_2 \oplus (e_1 e_2 e_3) g_3.$$

Lex–Algorithm

- Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = \oplus_{i=1}^{3} Eg_i$ with $f_1 = -3, f_2 = -1, f_3 = 2$. Let us consider the $[-3,2]$–sequence $H = (1,2,2,4,3,3,4,5,2,0)$.

H–degrees	$\mathbf{-3}$	-2	$\mathbf{-1}$	0	1	**2**	3	4	5	6	
H	(1,	2,	2,	4,	3,	3,	4,	5,	2,	0)	$-$
$\widetilde{H}_3$	(0,	0,	0,	0,	0,	**1**,	**4**,	**5**,	**2**,	**0**)	$-$
$\widetilde{H}_2$	(0,	0,	**1**,	**4**,	**3**,	**1**,	**0**,	0,	0,	0)	$-$
$\widetilde{H}_1$	(**1**,	**2**,	**1**,	**0**,	**0**,	0,	0,	0,	0,	0)	$=$
	(0,	0,	0,	0,	0,	**1**,	0,	0,	0,	0)	

- At the end, we do not obtain the null sequence 0_{10}, and so H is not a Hilbert sequence of a quotient of a free E-module. Indeed, one can observe that H does not satisfy the bound established: $3 = H(2) \not\leq \binom{4}{0} + \binom{3}{3} = 2$.

Next steps

- We intend to implement improvements to the *Macaulay2* package ExteriorModules. More precisely, given a submodule of F we would like to implement some algorithms
  - to compute the Generic Initial Module
  - to manage the Dual Module (in a general case)

  This problems are currently under investigation.

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Thanks