Minimal resolutions of graded modules over an exterior algebra

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Introduction

- Let K be a field, $E = K\langle e_1, \ldots, e_n\rangle$ the exterior algebra of a K-vector space V with basis $e_1, \ldots, e_n$.
- It is well–known that, even if E is not commutative, it behaves like a commutative local ring (**[Bruns and Herzog, 1996]**).
- We work on the category $\mathcal{M}$ of finitely generated $\mathbb{Z}$-graded left and right E-modules M.
- If $M \in \mathcal{M}$, we denote by $\beta_{i,j}(M)$ the graded Betti numbers of M and by $\mu_{i,j}(M)$ the graded Bass numbers of M, associate with **projective and injective minimal resolution** of M, respectively.
- Our aim is to give upper bounds for such invariants.

Introduction

- Ideas and techniques used in this work generalize the ones in **[Aramova et al., 1997], [Aramova et al., 1998]**. In those papers, in fact, the results concern ideals of an exterior algebra.
- Given a stable ideal $I \subset E$ one can deduce explicit formulas for the graded Betti numbers. These formulas allow a comparison of the graded Betti numbers of a stable ideal and its corresponding lexsegment ideal.
- Since the exterior algebra is self-dual, there exists a dual version to each of these theorems. So, one has analogous comparison of the graded Bass numbers of a stable ideal and its corresponding lexsegment ideal.
- A fundamental tool also for our pourpose is the class of **lexicographic submodules**. Such a class of monomial submodules has been deeply studied by the authors of this paper in **[Amata and Crupi, 2018b], [Amata and Crupi, 2018c]**. Some results on a particular class of graded E-modules has already published in **[Amata and Crupi, 2018a]**.

Exterior Algebra

Definitions

- Let K be a field. We denote by $E = K\langle e_1, \dots, e_n\rangle$ the exterior algebra of a K-vector space V with basis $e_1, \dots, e_n$.

- For any subset $\sigma = \{i_1, \dots, i_d\}$ of $\{1, \dots, n\}$, with $i_1 < i_2 < \dots < i_d$, we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \emptyset$.

- We put $fg = f \wedge g$ for any two elements f and g in E. An element $f \in E$ is called *homogeneous* of degree j if $f \in E_j$, where $E_j = \bigwedge^j V$.

- We define $\mathrm{supp}(e_\sigma) = \sigma = \{j : e_j \text{ divides } e_\sigma\}$ and $\mathrm{m}(e_\sigma) = \max\{i : i \in \mathrm{supp}(e_\sigma)\}$. Moreover, we set $\mathrm{m}(e_\sigma) = 0$ if $e_\sigma = 1$.

Example

Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$. If we consider the monomial $e_\sigma = e_1 e_2 e_4$, then $\mathrm{supp}(e_\sigma) = \{1, 2, 4\}$ and $\mathrm{m}(e_\sigma) = 4$.

Exterior Ideals

Definitions

- If I is a graded ideal in E, then the function $H_I : \mathbb{N} \to \mathbb{N}$ given by $H_I(d) = \dim_K I_d$ $(i \geq 0)$ is called the Hilbert function of I.
- Let I be a monomial ideal of E. I is called stable if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$.
- I is called strongly stable if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.
- Let $>_{\mathrm{lex}}$ the *lexicographic order* on the set of all monomials of degree $d \geq 1$ in E. A monomial ideal I of E is called a **lexsegment ideal** if for all monomials $u \in I$ and all monomials $v \in E$ with $\deg u = \deg v$ and $v >_{\mathrm{lex}} u$, then $v \in I$.

Exterior Ideals

Examples

- Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$. Consider the monomial ideal of E
$$I = (e_2e_3, e_3e_4e_5).$$
- The smallest stable ideal of E containing I is
$$Is = (e_1e_2, e_2e_3, e_1e_3e_4, e_3e_4e_5).$$
- The smallest strongly stable ideal of E containing Is is
$$Iss = (e_1e_2, e_1e_3, e_2e_3, e_1e_4e_5, e_2e_4e_5, e_3e_4e_5).$$
- The lexicographic ideal with the same Hilbert function of I is
$$I^{\text{lex}} = (e_1e_2, e_1e_3e_4).$$

Exterior Ideals

Considerations

If I is a graded ideal of E, then $\operatorname{Gin}(I)$ (generic initial ideal) is a strongly stable ideal of E with the same Hilbert function as I and, moreover, one has that $H_{E/I} = H_{E/\operatorname{Gin}(I)}$ and $\beta_{i,j}(E/I) \leq \beta_{i,j}(E/\operatorname{Gin}(I))$ for all i, j.
So we may assume I itself is a stable ideal without changing the Hilbert function.

Kruskal–Katona theorem

A formulation of this theorem can be done as follows:
Let $I \subset E$ be a graded ideal. Then $\beta_{0,j}(I) \leq \beta_{0,j}(I^{\text{lex}})$, for all j.

"Higher" Kruskal–Katona theorem

Let $I \subset E$ be a graded ideal. Then $\beta_{i,j}(I) \leq \beta_{i,j}(I^{\text{lex}})$, for all i, j.

Dual "Higher" Kruskal–Katona theorem

Let $I \subset E$ be a graded ideal. Then $\mu_{i,j}(E/I) \leq \mu_{i,j}(E/I^{\text{lex}})$, for all i, j.

Exterior Modules

Definitions

- For all $M \in \mathcal{M}$, the function $H_M : \mathbb{Z} \to \mathbb{Z}$ given by $H_M(d) = \dim_K M_d$ is called the Hilbert function of M.
- Let $F \in \mathcal{M}$ be a free module with homogeneous basis $g_1, \dots, g_r$, where $\deg(g_i) = f_i$, $i = 1, \dots, r$, with $f_1 \leq f_2 \leq \cdots \leq f_r$. Then $F = \oplus_{i=1}^r E g_i$.
- $M \in \mathcal{M}$ is monomial if M is a submodule generated by monomials of F: $M = I_1 g_1 \oplus \cdots \oplus I_r g_r$, with I_i a monomial ideal of E, for each i.
- A monomial submodule $M = \oplus_{i=1}^r I_i g_i$ of F is (strongly) stable if I_i is a (strongly) stable ideal of E, for each i, and $(e_1, \dots, e_n)^{f_{i+1}-f_i} I_{i+1} \subseteq I_i$, for $i = 1, \dots, r-1$.
- Let $>_{\mathrm{lex}_F}$ the POT extension in F of the *lexicographic order* $>_{\mathrm{lex}}$ in E. Let $\mathcal{L}$ be a monomial submodule of F. $\mathcal{L}$ is a **lexicographic submodule** if for all $u, v \in \mathrm{Mon}_d(F)$ with $u \in \mathcal{L}$ and $v >_{\mathrm{lex}_F} u$, one has $v \in \mathcal{L}$, for every $d \geq 1$.

Exterior Modules

Considerations

If M is a graded submodule of F, then $\mathrm{Gin}(M)$ (generic initial module) is a strongly stable submodule of F with the same Hilbert function as M and, moreover, one has that $H_{F/M} = H_{F/\mathrm{Gin}(M)}$ and $\beta_{i,j}(F/M) \leq \beta_{i,j}(F/\mathrm{Gin}(M))$ for all i, j.
So we may assume M itself is a stable submodule without changing the Hilbert function.

A generalization of Kruskal–Katona theorem

A formulation of this theorem can be done as follows:
Let $M \subset F$ be a graded module. Then

$$\beta_{0,j}(M) \leq \beta_{0,j}(M^{\mathrm{lex}}),$$

for all j.

Exterior Modules

Example

- Let F_d be the part of degree d of $F = \oplus_{i=1}^{r} Eg_i$ and denote by $\mathrm{Mon}_d(F)$ the set of all monomials of degree d of F.
- Let $E = K\langle e_1, e_2, e_3\rangle$ and $F = Eg_1 \oplus Eg_2$, with $\deg g_1 = 2$ and $\deg g_2 = 3$, the monomials of F, with respect to $>_{\mathrm{lex}_F}$, are ordered as follows:

$\mathrm{Mon}_2(F)$	g_1
$\mathrm{Mon}_3(F)$	$e_1g_1 >_{\mathrm{lex}_F} e_2g_1 >_{\mathrm{lex}_F} e_3g_1 >_{\mathrm{lex}_F} g_2$
$\mathrm{Mon}_4(F)$	$e_1e_2g_1 >_{\mathrm{lex}_F} e_1e_3g_1 >_{\mathrm{lex}_F} e_2e_3g_1 >_{\mathrm{lex}_F} e_1g_2 >_{\mathrm{lex}_F} e_2g_2 >_{\mathrm{lex}_F} e_3g_2$
$\mathrm{Mon}_5(F)$	$e_1e_2e_3g_1 >_{\mathrm{lex}_F} e_1e_2g_2 >_{\mathrm{lex}_F} e_1e_3g_2 >_{\mathrm{lex}_F} e_2e_3g_2$
$\mathrm{Mon}_6(F)$	$e_1e_2e_3g_2$

Exterior Modules

Example

Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = \oplus_{i=1}^{3} Eg_i$ with $f_1 = -2, f_2 = -1, f_3 = 1$. Consider the monomial submodule of F

$$M = (e_1e_3, e_1e_2e_4)g_1 \oplus (e_1e_2, e_2e_4, e_3e_4)g_2 \oplus (e_1e_2e_3, e_2e_3e_4)g_3.$$

The unique lexicographic submodule of F with the same Hilbert function of M

$$M^{\text{lex}} = (e_1e_2, e_1e_3e_4, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3, e_2e_3e_4)g_2 \oplus (e_1e_2e_3, e_1e_2e_4)g_3.$$

Maximal Betti numbers

Definitions

Let $M \in \mathcal{M}$, then M has a unique minimal graded free resolution over E:

$$F_\bullet : \ldots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

where $F_i = \oplus_j E(-j)^{\beta_{i,j}(M)}$. The integers $\beta_{i,j}(M)$ are called the graded Betti numbers of M.

Considerations

If $M = \oplus_{i=1}^r I_i g_i$ is an (almost) stable submodule of F, then we can use the *Aramova-Herzog-Hibi formula* for computing the graded Betti numbers of M:

$$\beta_{k,k+\ell}(M) = \sum_{i=1}^r \beta_{k,\,k+\ell}(I_i g_i) = \sum_{u \in G(M)_\ell} \binom{\mathrm{m}_F(u)+k-1}{\mathrm{m}_F(u)-1}, \qquad \text{for all } k.$$

Maximal Betti numbers

Considerations

Moreover, one can easily observe that

$$\sum_{u \in G(M)_\ell} \binom{\mathrm{m}_F(u)+k-1}{\mathrm{m}_F(u)-1} = \sum_{i=1}^{r} \left[\sum_{u \in G(I_i)_{\ell - f_\ell}} \binom{\mathrm{m}(u)+k-1}{\mathrm{m}(u)-1} \right].$$

A generalization of "higher" Kruskal–Katona Theorem

Some technical results yield the following result:
Let M be a graded submodule of F. Then

$$\beta_{i,j}(M) \leq \beta_{i,j}(M^{\mathrm{lex}}),$$

for all i, j.

Maximal Betti numbers

Example

Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = \oplus_{i=1}^{3} Eg_i$, $f_1 = -2, f_2 = -1, f_3 = 1$. Let

$$M = (e_1e_3, e_1e_2e_4)g_1 \oplus (e_1e_2, e_2e_4, e_3e_4)g_2 \oplus (e_1e_2e_3, e_2e_3e_4)g_3 \in \mathcal{M}.$$

We have a unique lexicographic module with the same Hilbert function of M:

$$M^{\text{lex}} = (e_1e_2, e_1e_3e_4, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3, e_2e_3e_4)g_2 \oplus (e_1e_2e_3, e_1e_2e_4)g_3.$$

total	7	21	44	78	125	187
0	1	2	3	4	5	6
1	4	12	25	44	70	104
2	–	–	–	–	–	–
3	–	–	–	–	–	–
4	2	7	16	30	50	77

Betti diagram for M

total	8	26	58	108	180	278
0	1	2	3	4	5	6
1	4	13	29	54	90	139
2	1	4	10	20	35	56
3	–	–	–	–	–	–
4	2	7	16	30	50	77

Betti diagram for M^{lex}

Graded Bass numbers

Definitions

- Let $M \in \mathcal{M}$, M has a unique minimal graded injective resolution:

$$I_\bullet : 0 \to M \to I^0 \to I^1 \to I^2 \to \dots ,$$

  where $I^i = \oplus_j E(n-j)^{\mu_{i,j}(M)}$. The integers $\mu_{i,j}(M)$ are called the graded Bass numbers of M.

- Let M^* be the right (left) E-module $\mathrm{Hom}_E(M,E)$. The duality between projective and injective resolutions implies the following relation between the graded Bass numbers of a module and the graded Betti numbers of its dual: $\beta_{i,j}(M) = \mu_{i,n-j}(M^*)$, for all i,j.

Considerations

If rank $F = 1$ with $f_1 = 0$, *i.e.*, $F = E$ and $M = I$ is a graded ideal of E, then $\mathrm{Hom}_E(E/I, E) \simeq 0 : I$, where $0 : I$ is the annihilator of I. If I is a lex ideal in E, then $0 : I$ is a lex ideal in E.

Graded Bass numbers

Considerations

Let us consider the dual module $\mathrm{Hom}_E(F/L, E)$, where $L = \oplus_{t=1}^r I_t g_t$ is lex submodule of F. Even though the annihilators above are lex ideals, the submodule $N = \oplus_{t=1}^r (0 : I_t) g_t$ is not a lex submodule of F. Conversely,

$$\widetilde{N} = (0 : I_3)g_1 \oplus (0 : I_2)g_2 \oplus (0 : I_1)g_3$$

is a lex submodule in F. Note that $(F/L)^* \simeq N \simeq \widetilde{N}$ as E–graded modules.

A generalization of dual "higher" Kruskal–Katona theorem

Let M be a graded submodule of E^r, $r \geq 1$. Then

$$\mu_{i,j}(E^r/M) \leq \mu_{i,j}(E^r/M^{\mathrm{lex}}),$$

for all i, j.

Maximal Bass numbers

Example

Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = E^3$. Consider the monomial submodule of F:

$$M = (e_1e_3, e_1e_2e_4)g_1 \oplus (e_1e_2, e_2e_4, e_3e_4)g_2 \oplus (e_1e_2e_3, e_2e_3e_4)g_3.$$

We have a unique lexicographic module with the same Hilbert function of M:

$$M^{\text{lex}} = (e_1e_2, e_1e_3, e_1e_4, e_2e_3)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2 \oplus (e_1e_2e_3)g_3.$$

total	3	9	23	46	80	127
0	–	–	–	–	–	–
1	–	6	19	41	74	120
2	3	3	4	5	6	7

Bass diagram for F/M

total	3	12	35	74	133	216
0	–	1	4	10	20	35
1	–	8	25	54	98	160
2	3	3	6	10	15	21

Bass diagram for F/M^{lex}

Maximal Bass numbers

Example

Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = \oplus_{i=1}^{3} Eg_i$, $f_1 = -2, f_2 = -1, f_3 = 1$. Let

$$M = (e_1e_3, e_1e_2e_4)g_1 \oplus (e_1e_2, e_2e_4, e_3e_4)g_2 \oplus (e_1e_2e_3, e_2e_3e_4)g_3 \in \mathcal{M}.$$

We have a unique lexicographic module with the same Hilbert function of M:

$$M^{\text{lex}} = (e_1e_2, e_1e_3e_4, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3, e_2e_3e_4)g_2 \oplus (e_1e_2e_3, e_1e_2e_4)g_3.$$

total	3	9	23	46	80	127
0	–	6	19	41	74	120
1	3	3	4	5	6	7

Bass diagram for F/M

total	3	12	35	74	133	216
0	–	10	32	70	128	210
1	3	2	3	4	5	6

Bass diagram for F/M^{lex}

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