Computational methods in "commutative algebra"

Luca Amata

Department of Mathematical and Computer Sciences,
Physical and Geological Sciences,
University of Messina

lamata@unime.it

Doctoral Day

June 15th, 2018

Introduction

- **Computer Algebra** is a subject of science devoted to methods for solving mathematically formulated problems by symbolic algorithms, and to implementation of these algorithms. It is based on the exact finite representation of mathematical objects and structures, and allows for symbolic and abstract manipulation by a computer.
- The interplay between computation and many areas of algebra is a natural phenomenon in view of the algorithmic character of the latter. The existence of inexpensive but powerful computational resources has enhanced these links by the opening up of many new areas of investigation in algebra.
- A frequent task in computational algebra is to certify that a given object has a certain property, also providing rather elaborate examples. Moreover, they have contributed to a new view of algorithmic methods not only as tools, but as new objects worthy of mathematical study. In fact an algorithmic approach to a classical problem may lead to a significant refinement of classical structure theory irrespective of algorithmic considerations.

Introduction

- **Computer Algebra** is a subject of science devoted to methods for solving mathematically formulated problems by symbolic algorithms, and to implementation of these algorithms. It is based on the exact finite representation of mathematical objects and structures, and allows for symbolic and abstract manipulation by a computer.
- The interplay between computation and many areas of algebra is a natural phenomenon in view of the algorithmic character of the latter. The existence of inexpensive but powerful computational resources has enhanced these links by the opening up of many new areas of investigation in algebra.
- A frequent task in computational algebra is to certify that a given object has a certain property, also providing rather elaborate examples. Moreover, they have contributed to a new view of algorithmic methods not only as tools, but as new objects worthy of mathematical study. In fact an algorithmic approach to a classical problem may lead to a significant refinement of classical structure theory irrespective of algorithmic considerations.

Introduction

- **Computer Algebra** is a subject of science devoted to methods for solving mathematically formulated problems by symbolic algorithms, and to implementation of these algorithms. It is based on the exact finite representation of mathematical objects and structures, and allows for symbolic and abstract manipulation by a computer.
- The interplay between computation and many areas of algebra is a natural phenomenon in view of the algorithmic character of the latter. The existence of inexpensive but powerful computational resources has enhanced these links by the opening up of many new areas of investigation in algebra.
- A frequent task in computational algebra is to certify that a given object has a certain property, also providing rather elaborate examples. Moreover, they have contributed to a new view of algorithmic methods not only as tools, but as new objects worthy of mathematical study. In fact an algorithmic approach to a classical problem may lead to a significant refinement of classical structure theory irrespective of algorithmic considerations.

Introduction

- **Extremal Betti numbers**
  - We implement in CoCoA[1] some algorithms in order to easy compute graded ideals of a polynomial ring with given extremal Betti numbers (positions as well as values).
  - More precisely, we develop a package for determining the conditions under which, given two positive integers n, r, $1 \leq r \leq n-1$, there exists a graded ideal of a polynomial ring in n variables with r extremal Betti numbers in the given position.
  - An algorithm to check whether an r-tuple of positive integers represents the admissible values of the r extremal Betti numbers is also described.

Introduction

- **Extremal Betti numbers**
  - We implement in CoCoA[1] some algorithms in order to easy compute graded ideals of a polynomial ring with given extremal Betti numbers (positions as well as values).
  - More precisely, we develop a package for determining the conditions under which, given two positive integers n, r, $1 \leq r \leq n-1$, there exists a graded ideal of a polynomial ring in n variables with r extremal Betti numbers in the given position.
  - An algorithm to check whether an r-tuple of positive integers represents the admissible values of the r extremal Betti numbers is also described.

Introduction

- **Extremal Betti numbers**
  - We implement in CoCoA[1] some algorithms in order to easy compute graded ideals of a polynomial ring with given extremal Betti numbers (positions as well as values).
  - More precisely, we develop a package for determining the conditions under which, given two positive integers n, r, $1 \leq r \leq n-1$, there exists a graded ideal of a polynomial ring in n variables with r extremal Betti numbers in the given position.
  - An algorithm to check whether an r-tuple of positive integers represents the admissible values of the r extremal Betti numbers is also described.

Introduction

- **Exterior Ideals**
  - We introduce a Macaulay2[2] package that allows one to deal with classes of monomial ideals over an exterior algebra E.
  - More precisely, we implement some algorithms in order to easily compute stable, strongly stable and lexsegment ideals in E.
  - Moreover, an algorithm to check whether an $(n+1)$-tuple $(1, h_1, \ldots, h_n)$ $(h_1 \leq n = \dim_K V)$ of nonnegative integers is the Hilbert function of a graded K–algebra of the form E/I, with I graded ideal of E, is given.
  - In particular, if $H_{E/I}$ is the Hilbert function of a graded K-algebra E/I, the package is able to construct the unique lexsegment ideal I^{lex} such that $H_{E/I} = H_{E/I^{\text{lex}}}$.
  - Finally, an algorithm to compute all the admissible Hilbert functions of graded K–algebras E/I, with given E, is also described.

Introduction

- **Exterior Ideals**
  - We introduce a Macaulay2[2] package that allows one to deal with classes of monomial ideals over an exterior algebra E.
  - More precisely, we implement some algorithms in order to easily compute stable, strongly stable and lexsegment ideals in E.
  - Moreover, an algorithm to check whether an $(n+1)$-tuple $(1, h_1, \ldots, h_n)$ $(h_1 \leq n = \dim_K V)$ of nonnegative integers is the Hilbert function of a graded K–algebra of the form E/I, with I graded ideal of E, is given.
  - In particular, if $H_{E/I}$ is the Hilbert function of a graded K-algebra E/I, the package is able to construct the unique lexsegment ideal I^{lex} such that $H_{E/I} = H_{E/I^{\text{lex}}}$.
  - Finally, an algorithm to compute all the admissible Hilbert functions of graded K–algebras E/I, with given E, is also described.

Introduction

- **Exterior Ideals**
  - We introduce a Macaulay2[2] package that allows one to deal with classes of monomial ideals over an exterior algebra E.
  - More precisely, we implement some algorithms in order to easily compute stable, strongly stable and lexsegment ideals in E.
  - Moreover, an algorithm to check whether an $(n+1)$-tuple $(1, h_1, \dots, h_n)$ ($h_1 \leq n = \dim_K V$) of nonnegative integers is the Hilbert function of a graded K–algebra of the form E/I, with I graded ideal of E, is given.
  - In particular, if $H_{E/I}$ is the Hilbert function of a graded K-algebra E/I, the package is able to construct the unique lexsegment ideal I^{lex} such that $H_{E/I} = H_{E/I^{\text{lex}}}$.
  - Finally, an algorithm to compute all the admissible Hilbert functions of graded K–algebras E/I, with given E, is also described.

Introduction

- **Exterior Ideals**
  - We introduce a Macaulay2[2] package that allows one to deal with classes of monomial ideals over an exterior algebra E.
  - More precisely, we implement some algorithms in order to easily compute stable, strongly stable and lexsegment ideals in E.
  - Moreover, an algorithm to check whether an $(n+1)$-tuple $(1, h_1, \dots, h_n)$ ($h_1 \leq n = \dim_K V$) of nonnegative integers is the Hilbert function of a graded K–algebra of the form E/I, with I graded ideal of E, is given.
  - In particular, if $H_{E/I}$ is the Hilbert function of a graded K-algebra E/I, the package is able to construct the unique lexsegment ideal I^{lex} such that $H_{E/I} = H_{E/I^{\mathrm{lex}}}$.
  - Finally, an algorithm to compute all the admissible Hilbert functions of graded K–algebras E/I, with given E, is also described.

Introduction

- **Exterior Ideals**
  - We introduce a Macaulay2[2] package that allows one to deal with classes of monomial ideals over an exterior algebra E.
  - More precisely, we implement some algorithms in order to easily compute stable, strongly stable and lexsegment ideals in E.
  - Moreover, an algorithm to check whether an $(n+1)$-tuple $(1, h_1, \ldots, h_n)$ $(h_1 \leq n = \dim_K V)$ of nonnegative integers is the Hilbert function of a graded K–algebra of the form E/I, with I graded ideal of E, is given.
  - In particular, if $H_{E/I}$ is the Hilbert function of a graded K-algebra E/I, the package is able to construct the unique lexsegment ideal I^{lex} such that $H_{E/I} = H_{E/I^{\text{lex}}}$.
  - Finally, an algorithm to compute all the admissible Hilbert functions of graded K–algebras E/I, with given E, is also described.

Extremal Betti numbers

- Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K.
- The **extremal Betti numbers**[3] of a graded ideal I of S are the non-zero top left corners in a block of zeroes in the Macaulay diagram of a graded free resolution of I.

- They are a refinement of the *Castelnuovo-Mumford regularity* and of the *projective dimension* of the ideal I.

Extremal Betti numbers

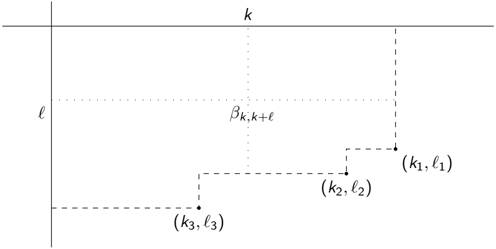
- Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K.
- The **extremal Betti numbers**[3] of a graded ideal I of S are the non-zero top left corners in a block of zeroes in the Macaulay diagram of a graded free resolution of I.

- They are a refinement of the *Castelnuovo-Mumford regularity* and of the *projective dimension* of the ideal I.

Extremal Betti numbers

- Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K.

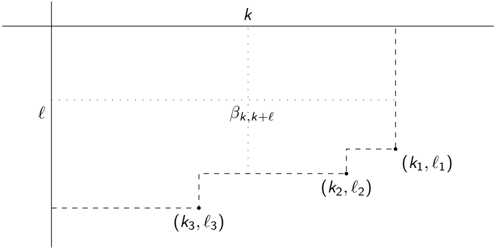
- The **extremal Betti numbers**[3] of a graded ideal I of S are the non-zero top left corners in a block of zeroes in the Macaulay diagram of a graded free resolution of I.



- They are a refinement of the *Castelnuovo-Mumford regularity* and of the *projective dimension* of the ideal I.

Extremal Betti numbers

- Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K.
- The **extremal Betti numbers**[3] of a graded ideal I of S are the non-zero top left corners in a block of zeroes in the Macaulay diagram of a graded free resolution of I.



- They are a refinement of the *Castelnuovo-Mumford regularity* and of the *projective dimension* of the ideal I.

Extremal Betti numbers

- **A question:**

  Given

  - two positive integers n, r, $1 \leq r \leq n-1$,
  - r pairs of positive integers (k_1, ℓ_1), …, (k_r, ℓ_r) such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$, $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$,
  - r positive integers $a_1, \ldots, a_r$,

  under which conditions does there exist a graded ideal I of S such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, …, $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

- **A positive answer:**

  - has been given in [4] and [5], when K is a field of characteristic zero,
  - the generic initial ideal, with respect to the graded reverse lexicographic order, of a graded ideal I of S is a strongly stable ideal,
  - the extremal Betti numbers of I, as well as their positions, are preserved by passing from I to his generic initial ideal.
  - Hence the problem can be reformulated in terms of strongly stable ideals.

Extremal Betti numbers

- **A question:**

  Given

  - two positive integers n, r, $1 \leq r \leq n-1$,
  - r pairs of positive integers (k_1, ℓ_1), ..., (k_r, ℓ_r) such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$, $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$,
  - r positive integers $a_1, \ldots, a_r$,

  under which conditions does there exist a graded ideal I of S such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, ..., $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

- **A positive answer:**

  - has been given in [4] and [5], when K is a field of characteristic zero,
  - the generic initial ideal, with respect to the graded reverse lexicographic order, of a graded ideal I of S is a strongly stable ideal,
  - the extremal Betti numbers of I, as well as their positions, are preserved by passing from I to his generic initial ideal.
  - Hence the problem can be reformulated in terms of strongly stable ideals.

Extremal Betti numbers

- **A question:**

  Given

  - two positive integers n, r, $1 \leq r \leq n-1$,
  - r pairs of positive integers (k_1, ℓ_1), ..., (k_r, ℓ_r) such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$, $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$,
  - r positive integers $a_1, \ldots, a_r$,

  under which conditions does there exist a graded ideal I of S such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, ..., $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

- **A positive answer:**

  - has been given in [4] and [5], when K is a field of characteristic zero,
  - the generic initial ideal, with respect to the graded reverse lexicographic order, of a graded ideal I of S is a strongly stable ideal,
  - the extremal Betti numbers of I, as well as their positions, are preserved by passing from I to his generic initial ideal.
  - Hence the problem can be reformulated in terms of strongly stable ideals.

Extremal Betti numbers

- **A question:**

  Given

  - two positive integers n, r, $1 \leq r \leq n-1$,
  - r pairs of positive integers (k_1, ℓ_1), ..., (k_r, ℓ_r) such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$, $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$,
  - r positive integers $a_1, \ldots, a_r$,

  under which conditions does there exist a graded ideal I of S such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, ..., $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

- **A positive answer:**

  - has been given in [4] and [5], when K is a field of characteristic zero,
  - the generic initial ideal, with respect to the graded reverse lexicographic order, of a graded ideal I of S is a strongly stable ideal,
  - the extremal Betti numbers of I, as well as their positions, are preserved by passing from I to his generic initial ideal.
  - Hence the problem can be reformulated in terms of strongly stable ideals.

Extremal Betti numbers

- **A question:**

  Given

  - two positive integers n, r, $1 \leq r \leq n-1$,
  - r pairs of positive integers (k_1, ℓ_1), ..., (k_r, ℓ_r) such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$, $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$,
  - r positive integers $a_1, \ldots, a_r$,

  under which conditions does there exist a graded ideal I of S such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, ..., $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

- **A positive answer:**

  - has been given in [4] and [5], when K is a field of characteristic zero,
  - the generic initial ideal, with respect to the graded reverse lexicographic order, of a graded ideal I of S is a strongly stable ideal,
  - the extremal Betti numbers of I, as well as their positions, are preserved by passing from I to his generic initial ideal.
  - Hence the problem can be reformulated in terms of strongly stable ideals.

Extremal Betti numbers

- **A question:**

  Given

  - two positive integers n, r, $1 \leq r \leq n-1$,
  - r pairs of positive integers (k_1, ℓ_1), ..., (k_r, ℓ_r) such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$, $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$,
  - r positive integers $a_1, \ldots, a_r$,

  under which conditions does there exist a graded ideal I of S such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, ..., $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

- **A positive answer:**

  - has been given in [4] and [5], when K is a field of characteristic zero,
  - the generic initial ideal, with respect to the graded reverse lexicographic order, of a graded ideal I of S is a strongly stable ideal,
  - the extremal Betti numbers of I, as well as their positions, are preserved by passing from I to his generic initial ideal.
  - Hence the problem can be reformulated in terms of strongly stable ideals.

Extremal Betti numbers

- **A question:**

  Given

  - two positive integers n, r, $1 \leq r \leq n-1$,
  - r pairs of positive integers (k_1, ℓ_1), ..., (k_r, ℓ_r) such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$, $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$,
  - r positive integers $a_1, \ldots, a_r$,

  under which conditions does there exist a graded ideal I of S such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, ..., $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

- **A positive answer:**

  - has been given in [4] and [5], when K is a field of characteristic zero,
  - the generic initial ideal, with respect to the graded reverse lexicographic order, of a graded ideal I of S is a strongly stable ideal,
  - the extremal Betti numbers of I, as well as their positions, are preserved by passing from I to his generic initial ideal.
  - Hence the problem can be reformulated in terms of strongly stable ideals.

Extremal Betti numbers

- **A question:**

  Given

  - two positive integers n, r, $1 \leq r \leq n-1$,
  - r pairs of positive integers (k_1, ℓ_1), ..., (k_r, ℓ_r) such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$, $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$,
  - r positive integers $a_1, \ldots, a_r$,

  under which conditions does there exist a graded ideal I of S such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, ..., $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

- **A positive answer:**

  - has been given in [4] and [5], when K is a field of characteristic zero,
  - the generic initial ideal, with respect to the graded reverse lexicographic order, of a graded ideal I of S is a strongly stable ideal,
  - the extremal Betti numbers of I, as well as their positions, are preserved by passing from I to his generic initial ideal.
  - Hence the problem can be reformulated in terms of strongly stable ideals.

Theorem[6]

Given two positive integers n, r, r pairs $(k_1, \ell_1), (k_2, \ell_2), \ldots, (k_r, \ell_r)$ and r positive integers $a_1, a_2, \ldots, a_r$ respecting the previous hypothesis and let K be a field of characteristic 0, then the following conditions are equivalent:

- there exists a strongly stable ideal $I \subsetneq S$, with extremal Betti numbers $\beta_{k_i, k_i+\ell_i}(I) = a_i$, for $i = 1, \ldots, r$;
- set $t = \max\{i : \ell_i \leq r - i\}$. The integers a_i satisfy the conditions:

  $$1 \leq a_i \leq |A_i \setminus \mathrm{LexShad}^{\ell_i - \ell_{i-1}}(A_{i-1})|, \quad \text{for} \quad i = 1, \ldots, r,$$

  where $A_0 = \emptyset$,
  - *(i)* $A_1 = \{u \in A(k_1, \ell_1) : u \geq_{\mathrm{lex}} x_{k_r-1} x_{k_1+1}\}$, whenever $\ell_1 = 2$;
  - *(ii)* $A_i = \{u \in A(k_i, \ell_i) : u \geq_{\mathrm{lex}} x_{k_r} x_{k_{r-1}} \cdots x_{k_{r-\ell_i+3}} x_{k_{r-\ell_i+2}-1} x_{k_i+1}\}$, for $i = 1, \ldots, t$, whenever $\ell_1 \geq 3$, and for $i = 2, \ldots, t$, whenever $\ell_1 = 2$;
  - *(iii)* $A_i = \{u \in A(k_i, \ell_i) : u \geq_{\mathrm{lex}} x_{k_r} x_{k_{r-1}} \cdots x_{k_{i+1}} x_{k_i+1}^{\ell_i-(r-i)}\}$, for $i = t+1, \ldots, r-1$;
  - *(iv)* $A_r = \{u \in A(k_r, \ell_r) : u \geq_{\mathrm{lex}} x_{k_r+1}^{\ell_r}\}$,

  and if $a_i = |[u, v]|$, with $u, v \in A_i$, then $1 \leq a_{i+1} \leq |A_{i+1} \setminus \mathrm{LexShad}^{\ell_{i+1}-\ell_i}([u, v])|$, for all $i = 1, \ldots r-1$,

with $2 < r \leq n-2$ (it has to be $n \geq 5$), $k_r \geq 2$, whenever $\ell_1 = 2$, and $1 \leq r \leq n-1$, $k_r \geq 1$, whenever $\ell_1 \geq 3$.

Theorem[6]

Given two positive integers n, r, r pairs $(k_1, \ell_1), (k_2, \ell_2), \ldots, (k_r, \ell_r)$ and r positive integers $a_1, a_2, \ldots, a_r$ respecting the previous hypothesis and let K be a field of characteristic 0, then the following conditions are equivalent:

- there exists a strongly stable ideal $I \subsetneq S$, with extremal Betti numbers $\beta_{k_i, k_i+\ell_i}(I) = a_i$, for $i = 1, \ldots, r$;
- set $t = \max\{i : \ell_i \leq r - i\}$. The integers a_i satisfy the conditions:

$$1 \leq a_i \leq |A_i \setminus \mathrm{LexShad}^{\ell_i - \ell_{i-1}}(A_{i-1})|, \quad \text{for} \quad i = 1, \ldots, r,$$

where $A_0 = \emptyset$,

(i) $A_1 = \{u \in A(k_1, \ell_1) : u \geq_{\mathrm{lex}} x_{k_r-1} x_{k_1+1}\}$, whenever $\ell_1 = 2$;

(ii) $A_i = \{u \in A(k_i, \ell_i) : u \geq_{\mathrm{lex}} x_{k_r} x_{k_{r-1}} \cdots x_{k_{r-\ell_i+3}} x_{k_{r-\ell_i+2}-1} x_{k_i+1}\}$, for $i = 1, \ldots, t$, whenever $\ell_1 \geq 3$, and for $i = 2, \ldots, t$, whenever $\ell_1 = 2$;

(iii) $A_i = \{u \in A(k_i, \ell_i) : u \geq_{\mathrm{lex}} x_{k_r} x_{k_{r-1}} \cdots x_{k_{i+1}} x_{k_i+1}^{\ell_i - (r-i)}\}$, for $i = t+1, \ldots, r-1$;

(iv) $A_r = \{u \in A(k_r, \ell_r) : u \geq_{\mathrm{lex}} x_{k_r+1}^{\ell_r}\}$,

and if $a_i = |[u, v]|$, with $u, v \in A_i$, then $1 \leq a_{i+1}$
$\leq |A_{i+1} \setminus \mathrm{LexShad}^{\ell_{i+1} - \ell_i}([u, v])|$, for all $i = 1, \ldots r - 1$,

with $2 < r \leq n - 2$ (it has to be $n \geq 5$), $k_r \geq 2$, whenever $\ell_1 = 2$, and $1 \leq r \leq n - 1$, $k_r \geq 1$, whenever $\ell_1 \geq 3$.

Theorem[6]

Given two positive integers n, r, r pairs $(k_1, \ell_1), (k_2, \ell_2), \ldots, (k_r, \ell_r)$ and r positive integers $a_1, a_2, \ldots, a_r$ respecting the previous hypothesis and let K be a field of characteristic 0, then the following conditions are equivalent:

- there exists a strongly stable ideal $I \subsetneq S$, with extremal Betti numbers $\beta_{k_i, k_i+\ell_i}(I) = a_i$, for $i = 1, \ldots, r$;
- set $t = \max\{i : \ell_i \leq r - i\}$. The integers a_i satisfy the conditions:

  $$1 \leq a_i \leq |A_i \setminus \mathrm{LexShad}^{\ell_i - \ell_{i-1}}(A_{i-1})|, \quad \text{for} \quad i = 1, \ldots, r,$$

  where $A_0 = \emptyset$,

  (i) $A_1 = \{u \in A(k_1, \ell_1) : u \geq_{\mathrm{lex}} x_{k_r-1}x_{k_1+1}\}$, whenever $\ell_1 = 2$;

  (ii) $A_i = \{u \in A(k_i, \ell_i) : u \geq_{\mathrm{lex}} x_{k_r}x_{k_{r-1}} \cdots x_{k_{r-\ell_i+3}}x_{k_{r-\ell_i+2}-1}x_{k_i+1}\}$, for $i = 1, \ldots, t$, whenever $\ell_1 \geq 3$, and for $i = 2, \ldots, t$, whenever $\ell_1 = 2$;

  (iii) $A_i = \{u \in A(k_i, \ell_i) : u \geq_{\mathrm{lex}} x_{k_r}x_{k_{r-1}} \cdots x_{k_{i+1}}x_{k_i+1}^{\ell_i-(r-i)}\}$, for $i = t+1, \ldots, r-1$;

  (iv) $A_r = \{u \in A(k_r, \ell_r) : u \geq_{\mathrm{lex}} x_{k_r+1}^{\ell_r}\}$,

  and if $a_i = |[u, v]|$, with $u, v \in A_i$, then $1 \leq a_{i+1}$ $\leq |A_{i+1} \setminus \mathrm{LexShad}^{\ell_{i+1}-\ell_i}([u, v])|$, for all $i = 1, \ldots r-1$,

with $2 < r \leq n-2$ (it has to be $n \geq 5$), $k_r \geq 2$, whenever $\ell_1 = 2$, and $1 \leq r \leq n-1$, $k_r \geq 1$, whenever $\ell_1 \geq 3$.

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
Loading CoCoA packages from directory:
  /cocoa-5.2/packages

With CoCoALib and external libraries GMP, Normaliz, Frobby
indent(VersionInfo()); -- for information about this version

/* */ use R::=QQ[x[1..7]];

/* */ Corners:=[[5,3],[4,4],[3,5],[1,7]];

/* */ a:=[1,2,5,1];

/* */ I:=StronglyStableIdealEB(R,Corners,a,1);

Strongly Stable Ideals has been computed:

Generators:  [x[1]^3, x[1]^2*x[2], x[1]^2*x[3], x[1]^2*x[4], x[1]^2*x[5], x[1]^2*x[6],
x[1]*x[2]^3, x[1]*x[2]^2*x[3], x[1]*x[2]^2*x[4], x[1]*x[2]^2*x[5], x[1]*x[2]*x[3]^2,
x[1]*x[2]*x[3]*x[4], x[1]*x[2]*x[3]*x[5], x[1]*x[2]*x[4]^3, x[1]*x[3]^4,
x[1]*x[3]^3*x[4], x[1]*x[3]^2*x[4]^2, x[1]*x[3]*x[4]^3, x[1]*x[4]^4, x[2]^7]

Borel Generators:  [x[1]^2*x[6], x[1]*x[2]*x[3]*x[5], x[1]*x[4]^4, x[2]^7]

-1\**-  *cocoa5*     All L10     (cocoa5-comint:run)-----------------------------------

-1\**-  Extremal_Betti.cocoa5    Top L1     (CoCoA5)-----------------------------------
```

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
Loading CoCoA packages from directory:
  /cocoa-5.2/packages

With CoCoALib and external libraries GMP, Normaliz, Frobby
indent(VersionInfo()); -- for information about this version

/* */ use R::=QQ[x[1..7]];

/* */ Corners:=[[5,3],[4,4],[3,5],[1,7]];

/* */ a:=[1,2,5,1];

/* */ I:=StronglyStableIdealEB(R,Corners,a,1);

Strongly Stable Ideals has been computed:

Generators:  [x[1]^3, x[1]^2*x[2], x[1]^2*x[3], x[1]^2*x[4], x[1]^2*x[5], x[1]^2*x[6],
x[1]*x[2]^3, x[1]*x[2]^2*x[3], x[1]*x[2]^2*x[4], x[1]*x[2]^2*x[5], x[1]*x[2]*x[3]^2,
x[1]*x[2]*x[3]*x[4], x[1]*x[2]*x[3]*x[5], x[1]*x[2]*x[4]^3, x[1]*x[3]^4,
x[1]*x[3]^3*x[4], x[1]*x[3]^2*x[4]^2, x[1]*x[3]*x[4]^3, x[1]*x[4]^4, x[2]^7]

Borel Generators:  [x[1]^2*x[6], x[1]*x[2]*x[3]*x[5], x[1]*x[4]^4, x[2]^7]
```

```
-1\**-  *cocoa5*     All L10    (cocoa5-comint:run)-----------------------------------
```

```
-1\**-  Extremal_Betti.cocoa5    Top L1     (CoCoA5)----------------------------------
```

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
Loading CoCoA packages from directory:
  /cocoa-5.2/packages

With CoCoALib and external libraries GMP, Normaliz, Frobby
indent(VersionInfo()); -- for information about this version

/* */ use R::=QQ[x[1..7]];
/* */ Corners:=[[5,3],[4,4],[3,5],[1,7]];
/* */ a:=[1,2,5,1];
/* */ I:=StronglyStableIdealEB(R,Corners,a,1);
Strongly Stable Ideals has been computed:
Generators:  [x[1]^3, x[1]^2*x[2], x[1]^2*x[3], x[1]^2*x[4], x[1]^2*x[5], x[1]^2*x[6],
x[1]*x[2]^3, x[1]*x[2]^2*x[3], x[1]*x[2]^2*x[4], x[1]*x[2]^2*x[5], x[1]*x[2]*x[3]^2,
x[1]*x[2]*x[3]*x[4], x[1]*x[2]*x[3]*x[5], x[1]*x[2]*x[4]^3, x[1]*x[3]^4,
x[1]*x[3]^3*x[4], x[1]*x[3]^2*x[4]^2, x[1]*x[3]*x[4]^3, x[1]*x[4]^4, x[2]^7]
Borel Generators:  [x[1]^2*x[6], x[1]*x[2]*x[3]*x[5], x[1]*x[4]^4, x[2]^7]
```

-1**- *cocoa5*   All L10   (cocoa5-comint:run)------------------------------

-1**- Extremal_Betti.cocoa5   Top L1   (CoCoA5)------------------------------

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
Loading CoCoA packages from directory:
  /cocoa-5.2/packages

With CoCoALib and external libraries GMP, Normaliz, Frobby
indent(VersionInfo()); -- for information about this version

/* */ use R::=QQ[x[1..7]];

/* */ Corners:=[[5,3],[4,4],[3,5],[1,7]];

/* */ a:=[1,2,5,1];

/* */ I:=StronglyStableIdealEB(R,Corners,a,1);

Strongly Stable Ideals has been computed:

Generators:  [x[1]^3, x[1]^2*x[2], x[1]^2*x[3], x[1]^2*x[4], x[1]^2*x[5], x[1]^2*x[6],
x[1]*x[2]^3, x[1]*x[2]^2*x[3], x[1]*x[2]^2*x[4], x[1]*x[2]^2*x[5], x[1]*x[2]*x[3]^2,
x[1]*x[2]*x[3]*x[4], x[1]*x[2]*x[3]*x[5], x[1]*x[2]*x[4]^3, x[1]*x[3]^4,
x[1]*x[3]^3*x[4], x[1]*x[3]^2*x[4]^2, x[1]*x[3]*x[4]^3, x[1]*x[4]^4, x[2]^7]

Borel Generators:  [x[1]^2*x[6], x[1]*x[2]*x[3]*x[5], x[1]*x[4]^4, x[2]^7]

-1\**-  *cocoa5*      All L10    (cocoa5-comint:run)------------------------------

-1\**-  Extremal_Betti.cocoa5    Top L1      (CoCoA5)----------------------------
```

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
Loading CoCoA packages from directory:
  /cocoa-5.2/packages

    ______    ______    ___       ______
   / ____/__ / ____/__ /   |     / ____/
  / /   / __`/ /   / __`/ /| |___/___ \
 / /___/ /_/ / /___/ /_/ / ___ /____/___/ /
 `____/`____/`____/`____/_/  |_|    /_____/

With CoCoALib and external libraries GMP, Normaliz, Frobby
indent(VersionInfo()); -- for information about this version

/* */ use R::=QQ[x[1..7]];

/* */ Corners:=[[5,3],[4,4],[3,5],[1,7]];

/* */ a:=[1,2,5,1];

/* */ I:=StronglyStableIdealEB(R,Corners,a,1);

Strongly Stable Ideals has been computed:

Generators:  [x[1]^3, x[1]^2*x[2], x[1]^2*x[3], x[1]^2*x[4], x[1]^2*x[5], x[1]^2*x[6],
x[1]*x[2]^3, x[1]*x[2]^2*x[3], x[1]*x[2]^2*x[4], x[1]*x[2]^2*x[5], x[1]*x[2]*x[3]^2,
x[1]*x[2]*x[3]*x[4], x[1]*x[2]*x[3]*x[5], x[1]*x[2]*x[4]^3, x[1]*x[3]^4,
x[1]*x[3]^3*x[4], x[1]*x[3]^2*x[4]^2, x[1]*x[3]*x[4]^3, x[1]*x[4]^4, x[2]^7]

Borel Generators:  [x[1]^2*x[6], x[1]*x[2]*x[3]*x[5], x[1]*x[4]^4, x[2]^7]

-1\**-  *cocoa5*    All L10    (cocoa5-comint:run)-----------------------------

-1\**-  Extremal_Betti.cocoa5   Top L1     (CoCoA5)-----------------------------
```

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
Piecewise Components:
L_1=[x[1]^3, x[1]^2*x[2], x[1]^2*x[3], x[1]^2*x[4], x[1]^2*x[5], x[1]^2*x[6]]
L_2=[x[1]^4, x[1]^3*x[2], x[1]^3*x[3], x[1]^3*x[4], x[1]^3*x[5], x[1]^2*x[2]^2,
x[1]^2*x[2]*x[3], x[1]^2*x[2]*x[4], x[1]^2*x[2]*x[5], x[1]^2*x[3]^2, x[1]^2*x[3]*x[4],
x[1]^2*x[3]*x[5], x[1]^2*x[4]^2, x[1]^2*x[4]*x[5], x[1]^2*x[5]^2, x[1]*x[2]^3,
x[1]*x[2]^2*x[3], x[1]*x[2]^2*x[4], x[1]*x[2]^2*x[5], x[1]*x[2]*x[3]^2,
x[1]*x[2]*x[3]*x[4], x[1]*x[2]*x[3]*x[5]]
L_3=[x[1]^5, x[1]^4*x[2], x[1]^4*x[3], x[1]^4*x[4], x[1]^3*x[2]^2, x[1]^3*x[2]*x[3],
x[1]^3*x[2]*x[4], x[1]^3*x[3]^2, x[1]^3*x[3]*x[4], x[1]^3*x[4]^2, x[1]^2*x[2]^3,
x[1]^2*x[2]^2*x[3], x[1]^2*x[2]^2*x[4], x[1]^2*x[2]*x[3]^2, x[1]^2*x[2]*x[3]*x[4],
x[1]^2*x[2]*x[4]^2, x[1]^2*x[3]^3, x[1]^2*x[3]^2*x[4], x[1]^2*x[3]*x[4]^2,
x[1]^2*x[4]^3, x[1]*x[2]^4, x[1]*x[2]^3*x[3], x[1]*x[2]^3*x[4], x[1]*x[2]^2*x[3]^2,
x[1]*x[2]^2*x[3]*x[4], x[1]*x[2]^2*x[4]^2, x[1]*x[2]*x[3]^3, x[1]*x[2]*x[3]^2*x[4],
x[1]*x[2]*x[3]*x[4]^2, x[1]*x[2]*x[4]^3, x[1]*x[3]^4, x[1]*x[3]^3*x[4],
x[1]*x[3]^2*x[4]^2, x[1]*x[3]*x[4]^3, x[1]*x[4]^4]
L_4=[x[1]^7, x[1]^6*x[2], x[1]^5*x[2]^2, x[1]^4*x[2]^3, x[1]^3*x[2]^4, x[1]^2*x[2]^5,
x[1]*x[2]^6, x[2]^7]
```

-1**- *cocoa5*   All L10   (cocoa5-comint:run)---------------------------------

-1**- Extremal_Betti.cocoa5   Top L1   (CoCoA5)-------------------------------

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
/* */ IsStronglyStable(I);
true
/* */ IsLexSegment(I);
false
/* */ PrintRes(I);
0 --> R[-8] --> R[-7]^6(+)R[-8]^2 --> R[-6]^15(+)R[-7]^10(+)R[-8]^5 -->
R[-5]^20(+)R[-6]^20(+)R[-7]^16 --> R[-4]^15(+)R[-5]^19(+)R[-6]^17(+)R[-8] -->
R[-3]^6(+)R[-4]^7(+)R[-5]^6(+)R[-7]
/* */ PrintBettiDiagram(I);
         0     1     2     3     4     5
- - - - - - - - - - - - - - - - - - - - - -
   3:    6    15    20    15     6     1
   4:    7    19    20    10     2     -
   5:    6    17    16     5     -     -
   6:    -     -     -     -     -     -
   7:    1     1     -     -     -     -
- - - - - - - - - - - - - - - - - - - - - -
 Tot:   20    52    56    30     8     1    /* */
```

```
-1\**-  *cocoa5*     All L10     (cocoa5-comint:run)--------------------------
```

```
-1\**-  Extremal_Betti.cocoa5    Top L1     (CoCoA5)--------------------------
```

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
/* */ IsStronglyStable(I);
true
/* */ IsLexSegment(I);
false
/* */ PrintRes(I);
0 --> R[-8] --> R[-7]^6(+)R[-8]^2 --> R[-6]^15(+)R[-7]^10(+)R[-8]^5 -->
R[-5]^20(+)R[-6]^20(+)R[-7]^16 --> R[-4]^15(+)R[-5]^19(+)R[-6]^17(+)R[-8] -->
R[-3]^6(+)R[-4]^7(+)R[-5]^6(+)R[-7]
/* */ PrintBettiDiagram(I);
         0     1     2     3     4    5
- - - - - - - - - - - - - - - - - - - -
   3:    6     15    20    15    6    1
   4:    7     19    20    10    2    -
   5:    6     17    16    5     -    -
   6:    -     -     -     -     -    -
   7:    1     1     -     -     -    -
- - - - - - - - - - - - - - - - - - - -
 Tot:    20    52    56    30    8    1    /* */
```

```
-1\**-  *cocoa5*    All L10    (cocoa5-comint:run)-----------------------------------
-1\**-  Extremal_Betti.cocoa5   Top L1    (CoCoA5)-----------------------------------
```

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
/* */ IsStronglyStable(I);
true
/* */ IsLexSegment(I);
false
/* */ PrintRes(I);
0 --> R[-8] --> R[-7]^6(+)R[-8]^2 --> R[-6]^15(+)R[-7]^10(+)R[-8]^5 -->
R[-5]^20(+)R[-6]^20(+)R[-7]^16 --> R[-4]^15(+)R[-5]^19(+)R[-6]^17(+)R[-8] -->
R[-3]^6(+)R[-4]^7(+)R[-5]^6(+)R[-7]
/* */ PrintBettiDiagram(I);
          0    1    2    3    4    5
- - - - - - - - - - - - - - - - - - -
    3:    6   15   20   15    6    1
    4:    7   19   20   10    2    -
    5:    6   17   16    5    -    -
    6:    -    -    -    -    -    -
    7:    1    1    -    -    -    -
- - - - - - - - - - - - - - - - - - -
  Tot:   20   52   56   30    8    1    /* */
```

```
-1\**-  *cocoa5*     All L10    (cocoa5-comint:run)-----------------------------
-1\**-  Extremal_Betti.cocoa5    Top L1     (CoCoA5)----------------------------
```

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
/* */ IsStronglyStable(I);
true
/* */ IsLexSegment(I);
false
/* */ PrintRes(I);
0 --> R[-8] --> R[-7]^6(+)R[-8]^2 --> R[-6]^15(+)R[-7]^10(+)R[-8]^5 -->
R[-5]^20(+)R[-6]^20(+)R[-7]^16 --> R[-4]^15(+)R[-5]^19(+)R[-6]^17(+)R[-8] -->
R[-3]^6(+)R[-4]^7(+)R[-5]^6(+)R[-7]
/* */ PrintBettiDiagram(I);
          0    1    2    3    4    5
- - - - - - - - - - - - - - - - - - -
    3:    6    15   20   15   6    1
    4:    7    19   20   10   2    -
    5:    6    17   16   5    -    -
    6:    -    -    -    -    -    -
    7:    1    1    -    -    -    -
- - - - - - - - - - - - - - - - - - -
  Tot:    20   52   56   30   8    1   /* */
```

-1**- *cocoa5* All L10 (cocoa5-comint:run)

-1**- Extremal_Betti.cocoa5 Top L1 (CoCoA5)

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
/* */ use R::=QQ[x[1..7]];
/* */ Corners:=[[5,3],[4,4],[3,5],[1,7]];
/* */ av:=AdmissibleValues(R,Corners);
/* */ av;
/* */ matrix(QQ,
[[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2],
 [1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 7, 1, 1, 2, 3],
 [1, 2, 3, 4, 5, 6, 7, 1, 2, 3, 4, 5, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 1, 1, 1, 2, 1, 1],
 [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]])
/* */
```

```
-1\**-  *cocoa5*    All L10    (cocoa5-comint:run)----------------------------------------
```

```
-1\**-  Extremal_Betti.cocoa5    Top L1    (CoCoA5)----------------------------------------
```

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
/* */ use R::=QQ[x[1..7]];
/* */ Corners:=[[5,3],[4,4],[3,5],[1,7]];
/* */ av:=AdmissibleValues(R,Corners);
/* */ av;
/* */ matrix(QQ,
[[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2],
 [1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 7, 1, 1, 2, 3],
 [1, 2, 3, 4, 5, 6, 7, 1, 2, 3, 4, 5, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 1, 1, 1, 2, 1, 1],
 [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]])
/* */
-1\**-  *cocoa5*    All L10    (cocoa5-comint:run)------------------------------------
-1\**-  Extremal_Betti.cocoa5   Top L1    (CoCoA5)-------------------------------------
```

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
/* */ use R::=QQ[x[1..7]];
/* */ Corners:=[[5,3],[4,4],[3,5],[1,7]];
/* */ av:=AdmissibleValues(R,Corners);
/* */ av;
/* */ matrix(QQ,
[[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2],
 [1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 7, 1, 1, 2, 3],
 [1, 2, 3, 4, 5, 6, 7, 1, 2, 3, 4, 5, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 1, 1, 1, 2, 1, 1],
 [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]])
/* */
```

-1**- *cocoa5* All L10 (cocoa5-comint:run)------------------------------------

-1**- Extremal_Betti.cocoa5 Top L1 (CoCoA5)------------------------------------

emacs@DISTOPICO

File Edit Options Buffers Tools CoCoA-5 Complete In/Out Signals Help

```
/* */ use R::=QQ[x[1..7]];

/* */ Corners:=[[5,3],[4,4],[3,5],[1,7]];

/* */ av:=AdmissibleValues(R,Corners);

/* */ av;

/* */ matrix(QQ,
[[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2],
 [1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 7, 1, 1, 2, 3],
 [1, 2, 3, 4, 5, 6, 7, 1, 2, 3, 4, 5, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 1, 1, 1, 2, 1, 1],
 [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]])

/* */
```

-1**- *cocoa5*   All L10   (cocoa5-comint:run)

-1**- Extremal_Betti.cocoa5   Top L1   (CoCoA5)

Exterior Ideals

- Let K be a field. We denote by $E = K\langle e_1, \ldots, e_n \rangle$ the exterior algebra of a K-vector space V with basis $e_1, \ldots, e_n$.
  - For any subset $\sigma = \{i_1, \ldots, i_d\}$ of $\{1, \ldots, n\}$, with $i_1 < i_2 < \cdots < i_d$, we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \emptyset$.
  - We define $\mathrm{supp}(e_\sigma) = \sigma = \{j \,:\, e_j \text{ divides } e_\sigma\}$ and $\mathrm{m}(e_\sigma) = \max\{i : i \in \mathrm{supp}(e_\sigma)\}$. Moreover, we set $\mathrm{m}(e_\sigma) = 0$ if $e_\sigma = 1$.
  - If I is a graded ideal in E, then the function $H_I : \mathbb{N} \to \mathbb{N}$ given by $H_I(d) = \dim_K I_d$ $(i \geq 0)$ is called the Hilbert function of I.
- Let I be a monomial ideal of E. I is called stable if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$.
- I is called strongly stable if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.

Exterior Ideals

- Let K be a field. We denote by $E = K\langle e_1, \ldots, e_n \rangle$ the exterior algebra of a K-vector space V with basis $e_1, \ldots, e_n$.
  - For any subset $\sigma = \{i_1, \ldots, i_d\}$ of $\{1, \ldots, n\}$, with $i_1 < i_2 < \cdots < i_d$, we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \emptyset$.
  - We define $\mathrm{supp}(e_\sigma) = \sigma = \{j : e_j \text{ divides } e_\sigma\}$ and $\mathrm{m}(e_\sigma) = \max\{i : i \in \mathrm{supp}(e_\sigma)\}$. Moreover, we set $\mathrm{m}(e_\sigma) = 0$ if $e_\sigma = 1$.
  - If I is a graded ideal in E, then the function $H_I : \mathbb{N} \to \mathbb{N}$ given by $H_I(d) = \dim_K I_d$ $(i \geq 0)$ is called the Hilbert function of I.
- Let I be a monomial ideal of E. I is called stable if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$.
- I is called strongly stable if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.

Exterior Ideals

- Let K be a field. We denote by $E = K\langle e_1, \ldots, e_n \rangle$ the exterior algebra of a K-vector space V with basis $e_1, \ldots, e_n$.
  - For any subset $\sigma = \{i_1, \ldots, i_d\}$ of $\{1, \ldots, n\}$, with $i_1 < i_2 < \cdots < i_d$, we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \emptyset$.
  - We define $\mathrm{supp}(e_\sigma) = \sigma = \{j \,:\, e_j \text{ divides } e_\sigma\}$ and $\mathrm{m}(e_\sigma) = \max\{i : i \in \mathrm{supp}(e_\sigma)\}$. Moreover, we set $\mathrm{m}(e_\sigma) = 0$ if $e_\sigma = 1$.
  - If I is a graded ideal in E, then the function $H_I : \mathbb{N} \to \mathbb{N}$ given by $H_I(d) = \dim_K I_d$ $(i \geq 0)$ is called the Hilbert function of I.
- Let I be a monomial ideal of E. I is called stable if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$.
- I is called strongly stable if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.

Exterior Ideals

- Let K be a field. We denote by $E = K\langle e_1, \ldots, e_n\rangle$ the exterior algebra of a K-vector space V with basis $e_1, \ldots, e_n$.
  - For any subset $\sigma = \{i_1, \ldots, i_d\}$ of $\{1, \ldots, n\}$, with $i_1 < i_2 < \cdots < i_d$, we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \emptyset$.
  - We define $\mathrm{supp}(e_\sigma) = \sigma = \{j : e_j \text{ divides } e_\sigma\}$ and $\mathrm{m}(e_\sigma) = \max\{i : i \in \mathrm{supp}(e_\sigma)\}$. Moreover, we set $\mathrm{m}(e_\sigma) = 0$ if $e_\sigma = 1$.
  - If I is a graded ideal in E, then the function $H_I : \mathbb{N} \to \mathbb{N}$ given by $H_I(d) = \dim_K I_d$ $(i \geq 0)$ is called the Hilbert function of I.
- Let I be a monomial ideal of E. I is called stable if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$.
- I is called strongly stable if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.

Exterior Ideals

- Let K be a field. We denote by $E = K\langle e_1, \ldots, e_n \rangle$ the exterior algebra of a K-vector space V with basis $e_1, \ldots, e_n$.
  - For any subset $\sigma = \{i_1, \ldots, i_d\}$ of $\{1, \ldots, n\}$, with $i_1 < i_2 < \cdots < i_d$, we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \emptyset$.
  - We define $\text{supp}(e_\sigma) = \sigma = \{j : e_j \text{ divides } e_\sigma\}$ and $\text{m}(e_\sigma) = \max\{i : i \in \text{supp}(e_\sigma)\}$. Moreover, we set $\text{m}(e_\sigma) = 0$ if $e_\sigma = 1$.
  - If I is a graded ideal in E, then the function $H_I : \mathbb{N} \to \mathbb{N}$ given by $H_I(d) = \dim_K I_d$ $(i \geq 0)$ is called the Hilbert function of I.
- Let I be a monomial ideal of E. I is called stable if for each monomial $e_\sigma \in I$ and each $j < \text{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\text{m}(e_\sigma)\}} \in I$.
- I is called strongly stable if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.

Exterior Ideals

- Let K be a field. We denote by $E = K\langle e_1, \ldots, e_n\rangle$ the exterior algebra of a K-vector space V with basis $e_1, \ldots, e_n$.
  - For any subset $\sigma = \{i_1, \ldots, i_d\}$ of $\{1, \ldots, n\}$, with $i_1 < i_2 < \cdots < i_d$, we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \emptyset$.
  - We define $\mathrm{supp}(e_\sigma) = \sigma = \{j \,:\, e_j \text{ divides } e_\sigma\}$ and $\mathrm{m}(e_\sigma) = \max\{i : i \in \mathrm{supp}(e_\sigma)\}$. Moreover, we set $\mathrm{m}(e_\sigma) = 0$ if $e_\sigma = 1$.
  - If I is a graded ideal in E, then the function $H_I : \mathbb{N} \to \mathbb{N}$ given by $H_I(d) = \dim_K I_d$ $(i \geq 0)$ is called the Hilbert function of I.
- Let I be a monomial ideal of E. I is called stable if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$.
- I is called strongly stable if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.

Exterior Ideals

- Let $>_{\text{lex}}$ the *lexicographic order* on the set of all monomials of degree $d \geq 1$ in E. A monomial ideal I of E is called a **lexsegment ideal** (lex ideal, for short) if for all monomials $u \in I$ and all monomials $v \in E$ with $\deg u = \deg v$ and $v >_{\text{lex}} u$, then $v \in I$.
- Let a and i be two positive integers. Then a has the unique i-th Macaulay expansion $a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$ with $a_i > a_{i-1} > \cdots a_j \geq j \geq 1$.
- We define $a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}$.
  We also set $0^{(i)} = 0$ for all $i \geq 1$.

Exterior Ideals

- Let $>_{\text{lex}}$ the *lexicographic order* on the set of all monomials of degree $d \geq 1$ in E. A monomial ideal I of E is called a **lexsegment ideal** (lex ideal, for short) if for all monomials $u \in I$ and all monomials $v \in E$ with $\deg u = \deg v$ and $v >_{\text{lex}} u$, then $v \in I$.
- Let a and i be two positive integers. Then a has the unique i-th Macaulay expansion $a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$ with $a_i > a_{i-1} > \cdots a_j \geq j \geq 1$.
- We define $a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}$. We also set $0^{(i)} = 0$ for all $i \geq 1$.

Exterior Ideals

- Let $>_{\text{lex}}$ the *lexicographic order* on the set of all monomials of degree $d \geq 1$ in E. A monomial ideal I of E is called a **lexsegment ideal** (lex ideal, for short) if for all monomials $u \in I$ and all monomials $v \in E$ with $\deg u = \deg v$ and $v >_{\text{lex}} u$, then $v \in I$.
- Let a and i be two positive integers. Then a has the unique i-th Macaulay expansion $a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$ with $a_i > a_{i-1} > \cdots a_j \geq j \geq 1$.
- We define $a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}$.
  We also set $0^{(i)} = 0$ for all $i \geq 1$.

Exterior Ideals

Thm Let $(h_1, \ldots, h_n)$ be a sequence of nonnegative integers. Then the following conditions are equivalent:

(a) $1 + \sum_{i=1}^{n} h_i t^i$ is the Hilbert series of a graded K–algebra E/I;

(b) $0 < h_{i+1} \leq h_i^{(i)}$, $0 < i \leq n-1$.

This theorem is known as the Kruskal–Katona theorem.

- If $(1, h_1, \ldots, h_n)$ is a sequence of nonnegative integers such that
  (i) $h_1 \leq n$,
  (ii) $0 < h_{i+1} \leq h_i^{(i)}$, $0 < i \leq n-1$,

  then there exists[7] a unique lex ideal I of an exterior algebra E with n generators over a field K such that $H_{E/I}(d) = h_d$ $(d = 0, \ldots, n)$.
- The sequence $(1, h_1, \ldots, h_n)$ is called the Hilbert sequence of E/I.

Exterior Ideals

Thm Let $(h_1, \ldots, h_n)$ be a sequence of nonnegative integers. Then the following conditions are equivalent:

(a) $1 + \sum_{i=1}^{n} h_i t^i$ is the Hilbert series of a graded K–algebra E/I;

(b) $0 < h_{i+1} \leq h_i^{(i)}$, $0 < i \leq n-1$.

This theorem is known as the Kruskal–Katona theorem.

- If $(1, h_1, \ldots, h_n)$ is a sequence of nonnegative integers such that
  - (i) $h_1 \leq n$,
  - (ii) $0 < h_{i+1} \leq h_i^{(i)}$, $0 < i \leq n-1$,

  then there exists[7] a unique lex ideal I of an exterior algebra E with n generators over a field K such that $H_{E/I}(d) = h_d$ $(d = 0, \ldots, n)$.

- The sequence $(1, h_1, \ldots, h_n)$ is called the Hilbert sequence of E/I.

Exterior Ideals

Thm Let $(h_1, \ldots, h_n)$ be a sequence of nonnegative integers. Then the following conditions are equivalent:

(a) $1 + \sum_{i=1}^{n} h_i t^i$ is the Hilbert series of a graded K–algebra E/I;

(b) $0 < h_{i+1} \leq h_i^{(i)}$, $0 < i \leq n-1$.

This theorem is known as the Kruskal–Katona theorem.

- If $(1, h_1, \ldots, h_n)$ is a sequence of nonnegative integers such that
  (i) $h_1 \leq n$,
  (ii) $0 < h_{i+1} \leq h_i^{(i)}$, $0 < i \leq n-1$,

  then there exists[7] a unique lex ideal I of an exterior algebra E with n generators over a field K such that $H_{E/I}(d) = h_d$ $(d = 0, \ldots, n)$.
- The sequence $(1, h_1, \ldots, h_n)$ is called the Hilbert sequence of E/I.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LL
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 :  loadPackage "ExteriorIdeals"
i2 :  E=QQ[e_1..e_5,SkewCommutative=>true]
i3 :  I=ideal {e_2*e_3,e_3*e_4*e_5}
o3 = ideal (e e , e e e )
             2 3   3 4 5
o3 :  Ideal of E
i4 :  Is=stableIdeal I
o4 = ideal (e e , e e e , e e , e e e )
             1 2   1 3 4   2 3   3 4 5
o4 :  Ideal of E
i5 :  Iss=stronglyStableIdeal Is
o5 = ideal (e e , e e , e e e , e e , e e e , e e e )
             1 2   1 3   1 4 5   2 3   2 4 5   3 4 5
o5 :  Ideal of E
i6 :  isStronglyStableIdeal Iss
o6 = true
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LL
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 :  loadPackage "ExteriorIdeals"

i2 :  E=QQ[e_1..e_5,SkewCommutative=>true]

i3 :  I=ideal {e_2*e_3,e_3*e_4*e_5}

o3 = ideal (e e , e e e )
             2 3   3 4 5

o3 :  Ideal of E

i4 :  Is=stableIdeal I

o4 = ideal (e e , e e e , e e , e e e )
             1 2   1 3 4   2 3   3 4 5

o4 :  Ideal of E

i5 :  Iss=stronglyStableIdeal Is

o5 = ideal (e e , e e , e e e , e e , e e e , e e e )
             1 2   1 3   1 4 5   2 3   2 4 5   3 4 5

o5 :  Ideal of E

i6 :  isStronglyStableIdeal Iss

o6 = true
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LL
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 :  loadPackage "ExteriorIdeals"

i2 :  E=QQ[e_1..e_5,SkewCommutative=>true]

i3 :  I=ideal {e_2*e_3,e_3*e_4*e_5}

o3 = ideal (e e , e e e )
             2 3   3 4 5

o3 :  Ideal of E

i4 :  Is=stableIdeal I

o4 = ideal (e e , e e e , e e , e e e )
             1 2   1 3 4   2 3   3 4 5

o4 :  Ideal of E

i5 :  Iss=stronglyStableIdeal Is

o5 = ideal (e e , e e , e e e , e e , e e e , e e e )
             1 2   1 3   1 4 5   2 3   2 4 5   3 4 5

o5 :  Ideal of E

i6 :  isStronglyStableIdeal Iss

o6 = true
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LL
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 :  loadPackage "ExteriorIdeals"

i2 :  E=QQ[e_1..e_5,SkewCommutative=>true]

i3 :  I=ideal {e_2*e_3,e_3*e_4*e_5}

o3 = ideal (e e , e e e )
             2 3   3 4 5

o3 :  Ideal of E

i4 :  Is=stableIdeal I

o4 = ideal (e e , e e e , e e , e e e )
             1 2   1 3 4   2 3   3 4 5

o4 :  Ideal of E

i5 :  Iss=stronglyStableIdeal Is

o5 = ideal (e e , e e , e e e , e e , e e e , e e e )
             1 2   1 3   1 4 5   2 3   2 4 5   3 4 5

o5 :  Ideal of E

i6 :  isStronglyStableIdeal Iss

o6 = true
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LL
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 :  loadPackage "ExteriorIdeals"

i2 :  E=QQ[e_1..e_5,SkewCommutative=>true]

i3 :  I=ideal {e_2*e_3,e_3*e_4*e_5}

o3 = ideal (e e , e e e )
             2 3   3 4 5

o3 :  Ideal of E

i4 :  Is=stableIdeal I

o4 = ideal (e e , e e e , e e , e e e )
             1 2   1 3 4   2 3   3 4 5

o4 :  Ideal of E

i5 :  Iss=stronglyStableIdeal Is

o5 = ideal (e e , e e , e e e , e e , e e e , e e e )
             1 2   1 3   1 4 5   2 3   2 4 5   3 4 5

o5 :  Ideal of E

i6 :  isStronglyStableIdeal Iss

o6 = true
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LL
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 :  loadPackage "ExteriorIdeals"

i2 :  E=QQ[e_1..e_5,SkewCommutative=>true]

i3 :  I=ideal {e_2*e_3,e_3*e_4*e_5}

o3 = ideal (e e , e e e )
             2 3   3 4 5
o3 :  Ideal of E

i4 :  Is=stableIdeal I

o4 = ideal (e e , e e e , e e , e e e )
             1 2   1 3 4   2 3   3 4 5
o4 :  Ideal of E

i5 :  Iss=stronglyStableIdeal Is

o5 = ideal (e e , e e , e e e , e e , e e e , e e e )
             1 2   1 3   1 4 5   2 3   2 4 5   3 4 5
o5 :  Ideal of E

i6 :  isStronglyStableIdeal Iss

o6 = true
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
i1 :  E=QQ[e_1..e_5,SkewCommutative=>true]
i2 :  isHilbertSequence({0,4,3,0,0,0},E)
o2 :  false
i3 :  lexIdeal({1,6,3,0,0,0,0},E)
stdio:24:1:(3):  error:  expected a Hilbert sequence
i4 :  lexIdeal({1,4,4},E)
o4 = ideal (e , e e , e e , e e e )
             1   2 3   2 4   3 4 5
o4 :  Ideal of E
i5 :  lexIdeal({1,5,7,4,0,0},E)
o5 = ideal (e e , e e , e e , e e e e )
             1 2   1 3   1 4   2 3 4 5
o5 :  Ideal of E
i6 :  I=ideal {e_2*e_3,e_2*e_4,e_2*e_5,e_1*e_3*e_4*e_5}
o6 = ideal (e e , e e , e e , e e e e )
             2 3   2 4   2 5   1 3 4 5
o6 :  Ideal of E
i7 :  hilbertSequence I
o7 = {1, 5, 7, 4, 0, 0}
o7 :  List
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
i1 :  E=QQ[e_1..e_5,SkewCommutative=>true]

i2 :  isHilbertSequence({0,4,3,0,0,0},E)
o2 :  false

i3 :  lexIdeal({1,6,3,0,0,0,0},E)
stdio:24:1:(3):  error:  expected a Hilbert sequence

i4 :  lexIdeal({1,4,4},E)

o4 = ideal (e , e e , e e , e e e )
             1   2 3   2 4   3 4 5

o4 :  Ideal of E
i5 :  lexIdeal({1,5,7,4,0,0},E)

o5 = ideal (e e , e e , e e , e e e e )
             1 2   1 3   1 4   2 3 4 5

o5 :  Ideal of E

i6 :  I=ideal {e_2*e_3,e_2*e_4,e_2*e_5,e_1*e_3*e_4*e_5}

o6 = ideal (e e , e e , e e , e e e e )
             2 3   2 4   2 5   1 3 4 5

o6 :  Ideal of E

i7 :  hilbertSequence I
o7 = {1, 5, 7, 4, 0, 0}
o7 :  List
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
i1 :  E=QQ[e_1..e_5,SkewCommutative=>true]

i2 :  isHilbertSequence({0,4,3,0,0,0},E)
o2 :  false

i3 :  lexIdeal({1,6,3,0,0,0,0},E)
stdio:24:1:(3):  error:  expected a Hilbert sequence

i4 :  lexIdeal({1,4,4},E)

o4 = ideal (e , e e , e e , e e e )
             1   2 3   2 4   3 4 5

o4 :  Ideal of E
i5 :  lexIdeal({1,5,7,4,0,0},E)

o5 = ideal (e e , e e , e e , e e e e )
             1 2   1 3   1 4   2 3 4 5

o5 :  Ideal of E

i6 :  I=ideal {e_2*e_3,e_2*e_4,e_2*e_5,e_1*e_3*e_4*e_5}

o6 = ideal (e e , e e , e e , e e e e )
             2 3   2 4   2 5   1 3 4 5

o6 :  Ideal of E

i7 :  hilbertSequence I
o7 = {1, 5, 7, 4, 0, 0}
o7 :  List
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
i1 :  E=QQ[e_1..e_5,SkewCommutative=>true]

i2 :  isHilbertSequence({0,4,3,0,0,0},E)
o2 :  false

i3 :  lexIdeal({1,6,3,0,0,0,0},E)
stdio:24:1:(3):  error:  expected a Hilbert sequence

i4 :  lexIdeal({1,4,4},E)

o4 = ideal (e , e e , e e , e e e )
             1   2 3   2 4   3 4 5
o4 :  Ideal of E
i5 :  lexIdeal({1,5,7,4,0,0},E)

o5 = ideal (e e , e e , e e , e e e e )
             1 2   1 3   1 4   2 3 4 5
o5 :  Ideal of E

i6 :  I=ideal {e_2*e_3,e_2*e_4,e_2*e_5,e_1*e_3*e_4*e_5}

o6 = ideal (e e , e e , e e , e e e e )
             2 3   2 4   2 5   1 3 4 5
o6 :  Ideal of E

i7 :  hilbertSequence I
o7 = {1, 5, 7, 4, 0, 0}
o7 :  List
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
i1 :  E=QQ[e_1..e_5,SkewCommutative=>true]

i2 :  isHilbertSequence({0,4,3,0,0,0},E)
o2 :  false

i3 :  lexIdeal({1,6,3,0,0,0,0},E)
stdio:24:1:(3):  error:  expected a Hilbert sequence

i4 :  lexIdeal({1,4,4},E)

o4 = ideal (e , e e , e e , e e e )
             1   2 3   2 4   3 4 5
o4 :  Ideal of E
i5 :  lexIdeal({1,5,7,4,0,0},E)

o5 = ideal (e e , e e , e e , e e e e )
             1 2   1 3   1 4   2 3 4 5
o5 :  Ideal of E

i6 :  I=ideal {e_2*e_3,e_2*e_4,e_2*e_5,e_1*e_3*e_4*e_5}

o6 = ideal (e e , e e , e e , e e e e )
             2 3   2 4   2 5   1 3 4 5
o6 :  Ideal of E

i7 :  hilbertSequence I
o7 = {1, 5, 7, 4, 0, 0}
o7 :  List
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
i1 :  E=QQ[e_1..e_5,SkewCommutative=>true]

i2 :  isHilbertSequence({0,4,3,0,0,0},E)
o2 :  false

i3 :  lexIdeal({1,6,3,0,0,0,0},E)
stdio:24:1:(3):  error:  expected a Hilbert sequence

i4 :  lexIdeal({1,4,4},E)

o4 = ideal (e , e e , e e , e e e )
             1   2 3   2 4   3 4 5
o4 :  Ideal of E
i5 :  lexIdeal({1,5,7,4,0,0},E)

o5 = ideal (e e , e e , e e , e e e e )
             1 2   1 3   1 4   2 3 4 5
o5 :  Ideal of E

i6 :  I=ideal {e_2*e_3,e_2*e_4,e_2*e_5,e_1*e_3*e_4*e_5}

o6 = ideal (e e , e e , e e , e e e e )
             2 3   2 4   2 5   1 3 4 5
o6 :  Ideal of E

i7 :  hilbertSequence I
o7 = {1, 5, 7, 4, 0, 0}
o7 :  List
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
i1 :  E=QQ[e_1..e_5,SkewCommutative=>true]

i2 :  isHilbertSequence({0,4,3,0,0,0},E)
o2 :  false

i3 :  lexIdeal({1,6,3,0,0,0,0},E)
stdio:24:1:(3):  error:  expected a Hilbert sequence

i4 :  lexIdeal({1,4,4},E)

o4 = ideal (e , e e , e e , e e e )
             1   2 3   2 4   3 4 5
o4 :  Ideal of E
i5 :  lexIdeal({1,5,7,4,0,0},E)

o5 = ideal (e e , e e , e e , e e e e )
             1 2   1 3   1 4   2 3 4 5
o5 :  Ideal of E

i6 :  I=ideal {e_2*e_3,e_2*e_4,e_2*e_5,e_1*e_3*e_4*e_5}

o6 = ideal (e e , e e , e e , e e e e )
             2 3   2 4   2 5   1 3 4 5
o6 :  Ideal of E

i7 :  hilbertSequence I
o7 = {1, 5, 7, 4, 0, 0}
o7 :  List
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
i1 :  E=QQ[e_1..e_4,SkewCommutative=>true]
i2 :  hilbSeqs=allHilbertSequences(E)
o2 :  {{1,4,6,4,1}, {1,4,6,4,0}, {1,4,6,3,0}, {1,4,6,2,0}, {1,4,6,1,0},
       --------------------------------------------------------------------
       {1,4,6,0,0}, {1,4,5,2,0}, {1,4,5,1,0}, {1,4,5,0,0}, {1,4,4,1,0},
       --------------------------------------------------------------------
       {1,4,4,0,0}, {1,4,3,1,0}, {1,4,3,0,0}, {1,4,2,0,0}, {1,4,1,0,0},
       --------------------------------------------------------------------
       {1,4,0,0,0}, {1,3,3,1,0}, {1,3,3,0,0}, {1,3,2,0,0}, {1,3,1,0,0},
       --------------------------------------------------------------------
       {1,3,0,0,0}, {1,2,1,0,0}, {1,2,0,0,0}, {1,1,0,0,0}, {1,0,0,0,0}}
o2 :  List
i3 :  transpose matrix hilbSeqs
o3 = | 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 |
     | 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 3 3 3 3 3 2 2 1 0 |
     | 6 6 6 6 6 6 5 5 5 4 4 3 3 2 1 0 3 3 2 1 0 1 0 0 0 |
     | 4 4 3 2 1 0 2 1 0 1 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 |
     | 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
             5       25
o3 :  Matrix ZZ <--- ZZ
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
i1 :  E=QQ[e_1..e_4,SkewCommutative=>true]
i2 :  hilbSeqs=allHilbertSequences(E)
o2 :  {{1,4,6,4,1}, {1,4,6,4,0}, {1,4,6,3,0}, {1,4,6,2,0}, {1,4,6,1,0},
       ------------------------------------------------------------------
       {1,4,6,0,0}, {1,4,5,2,0}, {1,4,5,1,0}, {1,4,5,0,0}, {1,4,4,1,0},
       ------------------------------------------------------------------
       {1,4,4,0,0}, {1,4,3,1,0}, {1,4,3,0,0}, {1,4,2,0,0}, {1,4,1,0,0},
       ------------------------------------------------------------------
       {1,4,0,0,0}, {1,3,3,1,0}, {1,3,3,0,0}, {1,3,2,0,0}, {1,3,1,0,0},
       ------------------------------------------------------------------
       {1,3,0,0,0}, {1,2,1,0,0}, {1,2,0,0,0}, {1,1,0,0,0}, {1,0,0,0,0}}
o2 :  List
i3 :  transpose matrix hilbSeqs
o3 = | 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 |
     | 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 3 3 3 3 3 2 2 1 0 |
     | 6 6 6 6 6 6 5 5 5 4 4 3 3 2 1 0 3 3 2 1 0 1 0 0 0 |
     | 4 4 3 2 1 0 2 1 0 1 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 |
     | 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
              5       25
o3 :  Matrix ZZ <--- ZZ
```

RESET INTERRUPT SAVE UPLOAD FILE

UT

n mathematics or

```
i1 :  E=QQ[e_1..e_4,SkewCommutative=>true]

i2 :  hilbSeqs=allHilbertSequences(E)

o2 :  {{1,4,6,4,1}, {1,4,6,4,0}, {1,4,6,3,0}, {1,4,6,2,0}, {1,4,6,1,0},
       ---------------------------------------------------------------
       {1,4,6,0,0}, {1,4,5,2,0}, {1,4,5,1,0}, {1,4,5,0,0}, {1,4,4,1,0},
       ---------------------------------------------------------------
       {1,4,4,0,0}, {1,4,3,1,0}, {1,4,3,0,0}, {1,4,2,0,0}, {1,4,1,0,0},
       ---------------------------------------------------------------
       {1,4,0,0,0}, {1,3,3,1,0}, {1,3,3,0,0}, {1,3,2,0,0}, {1,3,1,0,0},
       ---------------------------------------------------------------
       {1,3,0,0,0}, {1,2,1,0,0}, {1,2,0,0,0}, {1,1,0,0,0}, {1,0,0,0,0}}
o2 :  List

i3 :  transpose matrix hilbSeqs

o3 = | 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 |
     | 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 3 3 3 3 3 2 2 1 0 |
     | 6 6 6 6 6 6 5 5 5 4 4 3 3 2 1 0 3 3 2 1 0 1 0 0 0 |
     | 4 4 3 2 1 0 2 1 0 1 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 |
     | 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
              5       25
o3 :  Matrix ZZ <--- ZZ
```

2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

Next steps

- It would be interesting to implement a *Macaulay2* package to compute graded modules with given extremal Betti numbers[8].

- It would be useful to implement a *Macaulay2* package for monomial modules over an exterior algebra. More precisely, we would like to implement some algorithms to compute stable, strongly stable, lexsegment submodules in E and to classify the Hilbert functions of quotients of E. This problem is currently under investigation.

Next steps

- It would be interesting to implement a *Macaulay2* package to compute graded modules with given extremal Betti numbers[8].

- It would be useful to implement a *Macaulay2* package for monomial modules over an exterior algebra. More precisely, we would like to implement some algorithms to compute stable, strongly stable, lexsegment submodules in E and to classify the Hilbert functions of quotients of E. This problem is currently under investigation.

Bibliography I

- J. Abbott, A. M. Bigatti, and G. Lagorio, "CoCoA-5: a system for doing Computations in Commutative Algebra." Available at `http://cocoa.dima.unige.it`.
- D. R. Grayson and M. E. Stillman, "Macaulay2, a software system for research in algebraic geometry." Available at `http://www.math.uiuc.edu/Macaulay2/`.
- D. Bayer, H. Charalambous, and S. Popescu, "Extremal Betti numbers and applications to monomial ideals," *Journal of Algebra*, vol. 221, pp. 497–512, 1999.
- M. Crupi and R. Utano, "Extremal Betti numbers of graded ideals," *Results in Mathematics*, vol. 43, pp. 235–244, 05 2003.
- J. Herzog, L. Sharifan, and M. Varbaro, "The possible extremal Betti numbers of a homogeneous ideal," *Proceedings of the American Mathematical Society*, vol. 142, 2 2014.

Bibliography II

L. Amata and M. Crupi, "Computation of graded ideals with given extremal betti numbers in a polynomial ring," *Journal of Symbolic Computation*, 2018.

L. Amata and M. Crupi, "ExteriorIdeals: A package for computing monomial ideals in an exterior algebra." submitted to The Journal of Software for Algebra and Geometry, 2017.

M. Crupi, "Extremal Betti numbers of graded modules," *Journal of Pure and Applied Algebra*, vol. 220, pp. 2277–2288, 06 2016.

Thanks

LexShadow

- For $u, v \in \mathrm{Mon}_d(S)$, $u \geq_{\mathrm{lex}} v$, define the set

$$\mathcal{L}(u, v) = \{z \in \mathrm{Mon}_d(S) : u \geq_{\mathrm{lex}} z \geq_{\mathrm{lex}} v\}.$$

- We denote by $\min(\mathcal{M})$ the smallest monomial of $\mathcal{M}$ respect to $\geq_{\mathrm{lex}}$. Setting $w = \min(\mathcal{M})$, if $\ell > d$ is an integer, we define the set of monomials of degree ℓ in S called the *lexicographic shadow* of $\mathcal{M}$ as:

$$\mathrm{LexShad}^{\ell-d}(\mathcal{M}) = \mathcal{L}(x_1^{\ell}, w x_n^{\ell-d}).$$

- Given two positive integers k, d, with $1 \leq k < n$ and $d \geq 2$, we define the following set of monomials:

$$A(k, d) = \{u \in \mathrm{Mon}_d(S) : \mathrm{m}(u) = k + 1\}.$$

- If u, v, $u \geq_{\mathrm{lex}} v$, are two monomials of $A(k, d)$, we will denote by $[u, v]$ the subset of $A(k, d)$ defined as follows:

$$[u, v] = \{w \in A(k, d) : u \geq_{\mathrm{lex}} w \geq_{\mathrm{lex}} v\}.$$

Theorem

LexShadow

- For $u, v \in \mathrm{Mon}_d(S)$, $u \geq_{\text{lex}} v$, define the set
$$\mathcal{L}(u,v) = \{z \in \mathrm{Mon}_d(S) : u \geq_{\text{lex}} z \geq_{\text{lex}} v\}.$$
- We denote by $\min(\mathcal{M})$ the smallest monomial of $\mathcal{M}$ respect to $\geq_{\text{lex}}$. Setting $w = \min(\mathcal{M})$, if $\ell > d$ is an integer, we define the set of monomials of degree ℓ in S called the *lexicographic shadow* of $\mathcal{M}$ as:
$$\mathrm{LexShad}^{\ell-d}(\mathcal{M}) = \mathcal{L}(x_1^\ell, w x_n^{\ell-d}).$$
- Given two positive integers k, d, with $1 \leq k < n$ and $d \geq 2$, we define the following set of monomials:
$$A(k,d) = \{u \in \mathrm{Mon}_d(S) : \mathrm{m}(u) = k+1\}.$$
- If u, v, $u \geq_{\text{lex}} v$, are two monomials of $A(k,d)$, we will denote by $[u,v]$ the subset of $A(k,d)$ defined as follows:
$$[u,v] = \{w \in A(k,d) : u \geq_{\text{lex}} w \geq_{\text{lex}} v\}.$$

Theorem

LexShadow

- For $u, v \in \mathrm{Mon}_d(S)$, $u \geq_{\mathrm{lex}} v$, define the set

$$\mathcal{L}(u, v) = \{z \in \mathrm{Mon}_d(S) : u \geq_{\mathrm{lex}} z \geq_{\mathrm{lex}} v\}.$$

- We denote by $\min(\mathcal{M})$ the smallest monomial of $\mathcal{M}$ respect to $\geq_{\mathrm{lex}}$. Setting $w = \min(\mathcal{M})$, if $\ell > d$ is an integer, we define the set of monomials of degree ℓ in S called the *lexicographic shadow* of $\mathcal{M}$ as:

$$\mathrm{LexShad}^{\ell-d}(\mathcal{M}) = \mathcal{L}(x_1^{\ell}, w x_n^{\ell-d}).$$

- Given two positive integers k, d, with $1 \leq k < n$ and $d \geq 2$, we define the following set of monomials:

$$A(k, d) = \{u \in \mathrm{Mon}_d(S) : \mathrm{m}(u) = k + 1\}.$$

- If u, v, $u \geq_{\mathrm{lex}} v$, are two monomials of $A(k, d)$, we will denote by $[u, v]$ the subset of $A(k, d)$ defined as follows:

$$[u, v] = \{w \in A(k, d) : u \geq_{\mathrm{lex}} w \geq_{\mathrm{lex}} v\}.$$

Theorem

LexShadow

- For $u, v \in \mathrm{Mon}_d(S)$, $u \geq_{\mathrm{lex}} v$, define the set

$$\mathcal{L}(u, v) = \{z \in \mathrm{Mon}_d(S) : u \geq_{\mathrm{lex}} z \geq_{\mathrm{lex}} v\}.$$

- We denote by $\min(\mathcal{M})$ the smallest monomial of $\mathcal{M}$ respect to $\geq_{\mathrm{lex}}$. Setting $w = \min(\mathcal{M})$, if $\ell > d$ is an integer, we define the set of monomials of degree ℓ in S called the *lexicographic shadow* of $\mathcal{M}$ as:

$$\mathrm{LexShad}^{\ell-d}(\mathcal{M}) = \mathcal{L}(x_1^{\ell}, w x_n^{\ell-d}).$$

- Given two positive integers k, d, with $1 \leq k < n$ and $d \geq 2$, we define the following set of monomials:

$$A(k, d) = \{u \in \mathrm{Mon}_d(S) : \mathrm{m}(u) = k + 1\}.$$

- If u, v, $u \geq_{\mathrm{lex}} v$, are two monomials of $A(k, d)$, we will denote by $[u, v]$ the subset of $A(k, d)$ defined as follows:

$$[u, v] = \{w \in A(k, d) : u \geq_{\mathrm{lex}} w \geq_{\mathrm{lex}} v\}.$$

Theorem