Graded algebras:

theoretical and computational aspects

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Introduction

Introduction

- Classical problems in commutative algebra include the study of the **Hilbert functions** and **minimal graded free resolutions** of finitely generated graded modules over graded algebras. These topics represent important tools in algebraic geometry and are becoming increasingly important both in combinatorics and **computational algebra**.

- This dissertation aims to deepen the study of the above mentioned topics approaching some open problems in order to integrate the existing literature and developing some packages that can be useful in the framework of commutative algebra and algebraic geometry. All the algorithms presented in this thesis have been implemented and some of them are included in **Macaulay2** version 1.14.

Introduction

- Let K be a field. A K–algebra A is graded if $A = \bigoplus_{i \in \mathbb{Z}} A_i$ such that $A_i A_j \subset A_{i+j}$ ($KA_j \subset A_j$). The **graded** K-**algebras** we consider in this thesis are the standard polynomial ring $S = K[x_1, \ldots, x_n]$ and the exterior algebra $E = K\langle e_1, \ldots, e_n \rangle$ of a K-vector space with basis $e_1, \ldots, e_n$.

- Let $R \in \{S, E\}$. Our work environment is $\mathcal{M}$, the category of finitely generated $\mathbb{Z}$-graded left and right R-modules M, and we will denote by $F \in \mathcal{M}$ a finitely generated graded free module with homogeneous basis $g_1, \ldots, g_r$.

- In R one can introduce the notions of monomial and monomial ideal and therefore that of monomial submodule of F. More in details, a **monomial submodule** M of F is a submodule of the form $M = \oplus_{i=1}^r I_i g_i$, with I_i ($i = 1, \ldots, r$) monomial ideals in R, *i.e.*, ideals generated by monomials of R.

Preliminaries and notations

Exterior Algebra

Definitions

- Let K be a field. We denote by $E = K\langle e_1, \dots, e_n\rangle$ the **exterior algebra** of a K-vector space V with basis $e_1, \dots, e_n$.

- For any subset $\sigma = \{i_1, \dots, i_d\}$ of $\{1, \dots, n\}$, with $i_1 < i_2 < \cdots < i_d$, we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \emptyset$.

- We put $fg = f \wedge g$ for any two elements f and g in E. An element $f \in E$ is called *homogeneous* of degree j if $f \in E_j$, where $E_j = \bigwedge^j V$.

- We define $\mathrm{supp}(e_\sigma) = \sigma = \{j : e_j \text{ divides } e_\sigma\}$ and $\mathrm{m}(e_\sigma) = \max\{i : i \in \mathrm{supp}(e_\sigma)\}$. Moreover, we set $\mathrm{m}(e_\sigma) = 0$ if $e_\sigma = 1$.

Example
Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$. If we consider the monomial $e_\sigma = e_1 e_2 e_4$, then $\mathrm{supp}(e_\sigma) = \{1, 2, 4\}$ and $\mathrm{m}(e_\sigma) = 4$.

Exterior Ideals

Definitions

- If I is a graded ideal in E, then the function $H_I : \mathbb{Z} \to \mathbb{Z}$ given by $H_I(d) = \dim_K I_d$ $(i \geq 0)$ is called the **Hilbert function** of I.
- Let I be a monomial ideal of E. I is called **stable** if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$.
- I is called **strongly stable** if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.
- Let $>_{\mathrm{lex}}$ the *lexicographic order* on the set of all monomials of degree $d \geq 1$ in E. A monomial ideal I of E is called a **lexsegment ideal** if for all monomials $u \in I$ and all monomials $v \in E$ with $\deg u = \deg v$ and $v >_{\mathrm{lex}} u$, then $v \in I$.

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```
Macaulay2, version 1.14
with packages:  ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LLLBases,
                PrimaryDecomposition, ReesAlgebra, TangentCone, Truncations

i1 :  loadPackage "ExteriorIdeals";

i2 :  E=QQ[e_1..e_5,SkewCommutative=>true];

i3 :  I=ideal {e_2*e_3,e_3*e_4*e_5}
```

o3 = ideal $(e_2e_3\ ,\ e_3e_4e_5\ )$

```
o3 :  Ideal of E

i4 :  Is=stableIdeal I
```

o4 = ideal $(e_1e_2\ ,\ e_2e_3\ ,\ e_1e_3e_4\ ,\ e_3e_4e_5\ )$

```
o4 :  Ideal of E

i5 :  Iss=stronglyStableIdeal Is
```

o5 = ideal $(e_1e_2\ ,\ e_1e_3\ ,\ e_2e_3\ ,\ e_1e_4e_5\ ,\ e_2e_4e_5\ ,\ e_3e_4e_5\ )$

```
o5 :  Ideal of E

i6 :  isLexIdeal Iss
o6 = false
```

Exterior Modules

Definitions

- For all $M \in \mathcal{M}$, the function $H_M : \mathbb{Z} \to \mathbb{Z}$ given by $H_M(d) = \dim_K M_d$ is called the **Hilbert function** of M.

- Let $F \in \mathcal{M}$ be a free module with homogeneous basis $g_1, \ldots, g_r$, where $\deg(g_i) = f_i$, $i = 1, \ldots, r$, with $f_1 \leq f_2 \leq \cdots \leq f_r$. Then $F = \oplus_{i=1}^{r} E g_i$.

- $M \in \mathcal{M}$ is **monomial** if M is a submodule generated by monomials of F: $M = I_1 g_1 \oplus \cdots \oplus I_r g_r$, with I_i a monomial ideal of E.

- A monomial submodule $M = \oplus_{i=1}^{r} I_i g_i$ of F is **(strongly) stable** if I_i is a (strongly) stable ideal of E, for each i, and $(e_1, \ldots, e_n)^{f_{i+1} - f_i} I_{i+1} \subseteq I_i$, for $i = 1, \ldots, r-1$.

- Let $>_{\mathrm{lex}_F}$ the POT extension in F of the *lexicographic order* $>_{\mathrm{lex}}$ in E. Let $\mathcal{L}$ be a monomial submodule of F. $\mathcal{L}$ is a **lexicographic submodule** if for all $u, v \in \mathrm{Mon}_d(F)$ with $u \in \mathcal{L}$ and $v >_{\mathrm{lex}_F} u$, one has $v \in \mathcal{L}$, for every $d \geq 1$.

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```
i1 :  loadPackage "ExteriorModules";
i2 :  E=QQ[e_1..e_5,SkewCommutative=>true];
i3 :  F=E^2;
i4 :  I_1=ideal {e_1*e_2, e_1*e_3, e_1*e_4*e_5};
i5 :  I_2=ideal {e_1*e_2, e_2*e_3*e_4};
i6 :  M=createModule({I_1, I_2},F)
```

$$o6 = \text{image} \begin{vmatrix} e_1e_3 & e_1e_2 & e_1e_4e_5 & 0 & 0 \\ 0 & 0 & 0 & e_1e_2 & e_2e_3e_4 \end{vmatrix}$$

$$o6 : \text{E-module, submodule of } E^2$$

```
i7 :  Ms=stableModule M
```

$$o7 = \text{image} \begin{vmatrix} e_1e_2 & e_1e_3 & e_1e_4e_5 & e_2e_3e_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_1e_2 & e_2e_3e_4 \end{vmatrix}$$

$$o7 : \text{E-module, submodule of } E^2$$

```
i8 :  Mss=stronglyStableModule M
```

$$o8 = \text{image} \begin{vmatrix} e_1e_2 & e_1e_3 & e_1e_4e_5 & e_2e_3e_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_1e_2 & e_1e_3e_4 & e_2e_3e_4 \end{vmatrix}$$

$$o8 : \text{E-module, submodule of } E^2$$

Hilbert Functions

Hilbert Functions over exterior algebras

- The **Hilbert function** of a graded K-algebra computes the vector space dimension of its graded components. It encodes important information on the graded K-algebra.
- The **Macaulay**'s key idea about the existence of highly structured monomial ideals, the **lexicographic ideals**, which attain all Hilbert functions of quotients of polynomial rings, has revealed crucial in the polynomial ring context.
- The **Kruskal–Katona** theorem is the *squarefree* analogue of Macaulay's theorem and may be also interpreted as a theorem on Hilbert functions of quotients of exterior algebras in [Aramova et al., 1997].
- Macaulay's theorem was extended to modules by many authors, in particular by Hulett [Hulett, 1995] and Gasharov in [Gasharov, 1997].
- In this thesis we have focused our attention on graded modules over the exterior algebra.

Hilbert Functions over exterior algebras

- Assume M is a monomial submodule of $F = \oplus_{i=1}^{r} E g_i$.
  One can quickly verify that $H_F(d) = \dim_K F_d = 0$, for $d < f_1$ and $d > f_r + n$. Hence, it follows that

$$H_{F/M}(t) = \sum_{i=f_1}^{f_r+n} H_{F/M}(i) t^i,$$

  and we can associate to F/M the following sequence

$$(H_{F/M}(f_1), H_{F/M}(f_1+1), \ldots, H_{F/M}(f_r+n)) \in \mathbb{N}_0^{f_r+n-f_1+1}.$$

- Such a sequence is called the Hilbert sequence of F/M, and denoted by $Hs_{F/M}$ [Amata and Crupi, 2019c].
  The integers $f_1, f_1+1, \ldots, f_r+n$ are called the $Hs_{F/M}$-degrees.

Example

- Let F_d be the part of degree d of $F = \oplus_{i=1}^{r} Eg_i$ and denote by $\mathrm{Mon}_d(F)$ the set of all monomials of degree d of F.
- Let $E = K\langle e_1, e_2, e_3\rangle$ and $F = Eg_1 \oplus Eg_2$, with $\deg g_1 = 2$ and $\deg g_2 = 3$, the monomials of F, with respect to $>_{\mathrm{lex}_F}$, are ordered as follows:

$\mathrm{Mon}_2(F)$	g_1
$\mathrm{Mon}_3(F)$	$e_1g_1 >_{\mathrm{lex}_F} e_2g_1 >_{\mathrm{lex}_F} e_3g_1 >_{\mathrm{lex}_F} g_2$
$\mathrm{Mon}_4(F)$	$e_1e_2g_1 >_{\mathrm{lex}_F} e_1e_3g_1 >_{\mathrm{lex}_F} e_2e_3g_1 >_{\mathrm{lex}_F} e_1g_2 >_{\mathrm{lex}_F} e_2g_2 >_{\mathrm{lex}_F} e_3g_2$
$\mathrm{Mon}_5(F)$	$e_1e_2e_3g_1 >_{\mathrm{lex}_F} e_1e_2g_2 >_{\mathrm{lex}_F} e_1e_3g_2 >_{\mathrm{lex}_F} e_2e_3g_2$
$\mathrm{Mon}_6(F)$	$e_1e_2e_3g_2$

Hilbert Functions over exterior algebras

- Let M be a graded submodule of $F = \oplus_{i=1}^{r} Eg_i$ and let $H_{F/M}$ the Hilbert function of F/M. There exists an integer $N \le r$ such that we have the unique expression

$$H_{F/M}(d) = \sum_{i=N+1}^{r} \binom{n}{d-f_i} + a,$$

where

$$a = \binom{a_0}{d-f_N} + \binom{a_1}{d-f_N-1} + \cdots + \binom{a_s}{d-f_N-s} < \binom{n}{d-f_N}$$

is the Macaulay representation of a lex_F segment in degree $d-f_N$ in the N-th component of F.

- Moreover,

$$H_{F/M}(d+1) \le \sum_{i=N+1}^{r} \binom{n}{d-f_i+1} + a^{(d-f_N)},$$

for $d \ge \operatorname{indeg} Hs_{F/M} + 1$.

A generalization of Kruskal–Katona's Theorem

Let $(f_1, f_2, \dots, f_r) \in \mathbb{Z}^r$ be an r–tuple such that $f_1 \le f_2 \le \cdots \le f_r$ and let $(h_{f_1}, h_{f_1+1}, \dots, h_{f_r+n})$ be a sequence of nonnegative integers.

Set $s = \min\{k \in [f_1, f_r + n] : h_k \ne 0\}$,

and $\tilde{r}_j = |\{p \in [r] : f_p = s + j\}|, \quad \text{for} \quad j = 0, 1.$

Then the following conditions are equivalent:

(a) $\sum_{i=s}^{f_r+n} h_i t^i$ is the Hilbert series of a graded E-module F/M, with $F = \oplus_{i=1}^{r} E g_i$ finitely generated graded free E-module with the basis elements g_i of degrees f_i;

(b) $h_s \le \tilde{r}_0$, $h_{s+1} \le n\tilde{r}_0 + \tilde{r}_1$, $h_i = \sum_{j=N+1}^{r} \binom{n}{i-f_j} + a$, where a is a positive integer less than $\binom{n}{i-f_N}$, $0 < N \le r$, and $h_{i+1} \le \sum_{j=N+1}^{r} \binom{n}{i-f_j+1} + a^{(i-f_N)}$, $i = s+1, \dots, f_r + n$;

(c) there exists a unique lexicographic submodule L of a finitely generated graded free E-module $F = \oplus_{i=1}^{r} E g_i$ with the basis elements g_i of degrees f_i and such that $\sum_{i=s}^{f_r+n} h_i t^i$ is the Hilbert series of F/L.

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```
i1 :  loadPackage "ExteriorModules";
i2 :  E=QQ[e_1..e_4,SkewCommutative=>true];
i3 :  F=E^3;
i4 :  I_1=ideal {e_1*e_2, e_3*e_4};
i5 :  I_2=ideal {e_1*e_2, e_2*e_3*e_4};
i6 :  I_3=ideal {e_2*e_3*e_4};
i7 :  M=createModule({I_1, I_2, I_3},F)
o7 = image | e_1e_2 e_3e_4 0      0        0        |
           | 0      0      e_1e_2 e_2e_3e_4 0        |
           | 0      0      0      0        e_2e_3e_4 |
                               3
o7 :  E-module, submodule of E
i8 :  L=lexModule M
o8 = image | e_1e_2 e_1e_3 e_1e_4 e_2e_3e_4 0         0         0         0         0            |
           | 0      0      0      0         e_1e_2e_3 e_1e_2e_4 e_1e_3e_4 e_2e_3e_4 0            |
           | 0      0      0      0         0         0         0         0         e_1e_2e_3e_4 |
                               3
o8 :  E-module, submodule of E
i9 :  hilbertSequence M
o9 = {3, 12, 15, 4, 0}
09 :  List
```

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```
i1 : loadPackage "ExteriorModules";
i2 : E=QQ[e_1..e_4,SkewCommutative=>true];
i3 : F=E^{2,0,-2};
i4 : I_1=ideal {e_1*e_2, e_3*e_4};
i5 : I_2=ideal {e_1*e_2, e_2*e_3*e_4};
i6 : I_3=ideal {e_2*e_3*e_4};
i7 : M=createModule({I_1, I_2, I_3},F)
```

$$o7 = \text{image} \begin{vmatrix} e\_1e\_2 & e\_3e\_4 & 0 & 0 & 0 \\ 0 & 0 & e\_1e\_2 & e\_2e\_3e\_4 & 0 \\ 0 & 0 & 0 & 0 & e\_2e\_3e\_4 \end{vmatrix}$$

```
                              3
o7 : E-module, submodule of E
i8 : L=lexModule M
```

$$o8 = \text{image} \begin{vmatrix} e\_1e\_2 & e\_1e\_3 & e\_2e\_3e\_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & e\_1e\_2 & e\_1e\_3e\_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & e\_1e\_2e\_3 \end{vmatrix}$$

```
                              3
o8 : E-module, submodule of E
i9 : hilbertSequence M
o9 = {1, 4, 5, 4, 6, 5, 6, 3, 0}
09 : List
```

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```
i1 :  loadPackage "ExteriorModules";
i2 :  E=QQ[e_1..e_4,SkewCommutative=>true];
i3 :  F=E^3;
i4 :  hs={3, 12, 15, 4, 0};
i5 :  lexModule(hs,F)
o5 = image | e_1e_2 e_1e_3 e_1e_4 e_2e_3e_4 0        0        0        0        0          |
           | 0      0      0      0        e_1e_2e_3 e_1e_2e_4 e_1e_3e_4 e_2e_3e_4 0        |
           | 0      0      0      0        0        0        0        0        e_1e_2e_3e_4 |
                                   3
o5 :  E-module, submodule of E
i6 :  F=E^{2,0,-2};
i7 :  hs={1, 4, 5, 4, 6, 5, 6, 3, 0};
i8 :  lexModuleBySequences(hs,F)
o8 = image | e_1e_2 e_1e_3 e_2e_3e_4 0      0        0         |
           | 0      0      0        e_1e_2 e_1e_3e_4 0         |
           | 0      0      0        0      0        e_1e_2e_3 |
                                   3
o8 :  E-module, submodule of E
i9 :  F=E^{3,1,-2};
i10 :  hs={1, 2, 2, 4, 3, 3, 4, 5, 2, 0};
i11 :  isHilbertSequence(hs,F)
o11 = false
```

Lex–Algorithm

- Let $E = K\langle e_1, e_2, e_3, e_4\rangle$, $F = E^3$ and let us consider the sequence $H = (3, 12, 15, 4, 0)$.

H–degrees	0	1	2	3	4	
H	(3,	12,	15,	4,	0)	$-$
Hs_{E/I_3}	(1,	4,	6,	4,	0)	$-$
Hs_{E/I_2}	(1,	4,	6,	0,	0)	$-$
Hs_{E/I_1}	(1,	4,	3,	0,	0)	$=$
0_5	(0,	0,	0,	0,	0)	

- $M^{\text{lex}} = \oplus_i^r I_i g_i$ is the unique lex submodule with Hilbert sequence H.

$$M^{\text{lex}} = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2 \oplus (e_1e_2e_3e_4)g_3.$$

Lex–Algorithm

- Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = \oplus_{i=1}^{3} Eg_i$ with $f_1 = -2, f_2 = 0, f_3 = 2$. Let us consider the $[-2, 6]$–sequence $H = (1, 4, 5, 4, 6, 5, 6, 3, 0)$.

H–degrees	-2	-1	0	1	2	3	4	5	6	
H	(1,	4,	5,	4,	6,	5,	6,	3,	0)	$-$
$\widetilde{H}_3$	(0,	0,	0,	0,	1,	4,	6,	3,	0)	$-$
$\widetilde{H}_2$	(0,	0,	1,	4,	5,	1,	0,	0,	0)	$-$
$\widetilde{H}_1$	(1,	4,	4,	0,	0,	0,	0,	0,	0)	$=$
0_9	(0,	0,	0,	0,	0,	0,	0,	0,	0)	

- $M^{\text{lex}} = \oplus_i^r I_i g_i$ is the unique lex submodule with Hilbert sequence H.

$$M^{\text{lex}} = (e_1e_2, e_1e_3, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3e_4)g_2 \oplus (e_1e_2e_3)g_3.$$

Lex–Algorithm

- Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = \oplus_{i=1}^{3} Eg_i$ with $f_1 = -3, f_2 = -1, f_3 = 2$. Let us consider the $[-3, 2]$–sequence $H = (1, 2, 2, 4, 3, 3, 4, 5, 2, 0)$.

H–degrees	−3	−2	−1	0	1	2	3	4	5	6	
H	(1,	2,	2,	4,	3,	3,	4,	5,	2,	0)	−
$\widetilde{H}_3$	(0,	0,	0,	0,	0,	1,	4,	5,	2,	0)	−
$\widetilde{H}_2$	(0,	0,	1,	4,	3,	1,	0,	0,	0,	0)	−
$\widetilde{H}_1$	(1,	2,	1,	0,	0,	0,	0,	0,	0,	0)	=
	(0,	0,	0,	0,	0,	1,	0,	0,	0,	0)	

- At the end, we do not obtain the null sequence 0_{10}, and so H is not a Hilbert sequence of a quotient of a free E-module. Indeed, one can observe that H does not satisfy the bound established: $3 = H(2) \nleq \binom{4}{0} + \binom{3}{3} = 2$.

Minimal Resolutions

Maximal Betti Numbers

- Regarding **minimal graded resolutions**, many authors have been interested in the problem of giving upper bounds for the graded Betti numbers of graded submodules in $\mathcal{M}$.
- The results of Bigatti [Bigatti, 1993], Hulett [Hulett, 1993] and Pardue [Pardue, 1994, Pardue, 1996] show that among all graded submodules of a free module over S with a given Hilbert function, the lexicographic submodule has the largest graded Betti numbers. Aramova, Herzog and Hibi in [Aramova et al., 1997] have studied ideals of an exterior algebra.
- In this dissertation we will show that similar results also hold in the exterior algebra context.

Maximal Betti numbers

Definitions
Let $M \in \mathcal{M}$, then M has a unique **minimal graded free resolution** over E:

$$F_\bullet : \ldots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

where $F_i = \oplus_j E(-j)^{\beta_{i,j}(M)}$. The integers $\beta_{i,j}(M)$ are called the **graded Betti numbers** of M.

Considerations
If $M = \oplus_{i=1}^r I_i g_i$ is a stable submodule of F, then we can use the *Aramova-Herzog-Hibi formula* for computing the graded Betti numbers of M:

$$\beta_{k,k+\ell}(M) = \sum_{i=1}^{r} \beta_{k,\,k+\ell}(I_i g_i) = \sum_{u \in G(M)_\ell} \binom{\mathrm{m}_F(u)+k-1}{\mathrm{m}_F(u)-1}, \qquad \text{for all } k.$$

Maximal Betti numbers

Considerations
Moreover, one can easily observe that

$$\sum_{u \in G(M)_\ell} \binom{\mathrm{m}_F(u)+k-1}{\mathrm{m}_F(u)-1} = \sum_{i=1}^{r} \left[\sum_{u \in G(I_i)_{\ell - f_\ell}} \binom{\mathrm{m}(u)+k-1}{\mathrm{m}(u)-1} \right].$$

A generalization of "higher" Kruskal–Katona Theorem
Some technical results yield the following result:
Let M be a graded submodule of F. Then

$$\beta_{i,j}(M) \leq \beta_{i,j}(M^{\mathrm{lex}}),$$

for all i, j.

Maximal Betti numbers

Example

Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = \oplus_{i=1}^{3} Eg_i$, $f_1 = -2, f_2 = -1, f_3 = 1$. Let

$$M = (e_1e_3, e_1e_2e_4)g_1 \oplus (e_1e_2, e_2e_4, e_3e_4)g_2 \oplus (e_1e_2e_3, e_2e_3e_4)g_3 \in \mathcal{M}.$$

We have a unique lexicographic module with the same Hilbert function of M:

$$M^{\text{lex}} = (e_1e_2, e_1e_3e_4, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3, e_2e_3e_4)g_2 \oplus (e_1e_2e_3, e_1e_2e_4)g_3.$$

total	7	21	44	78	125	187
0	1	2	3	4	5	6
1	4	12	25	44	70	104
2	—	—	—	—	—	—
3	—	—	—	—	—	—
4	2	7	16	30	50	77

Betti diagram for M

total	8	26	58	108	180	278
0	1	2	3	4	5	6
1	4	13	29	54	90	139
2	1	4	10	20	35	56
3	—	—	—	—	—	—
4	2	7	16	30	50	77

Betti diagram for M^{lex}

Graded Bass numbers

Definitions

- Let $M \in \mathcal{M}$, M has a unique **minimal graded injective resolution**:

$$I_\bullet : 0 \to M \to I^0 \to I^1 \to I^2 \to \dots,$$

  where $I^i = \oplus_j E(n-j)^{\mu_{i,j}(M)}$. The integers $\mu_{i,j}(M)$ are called the **graded Bass numbers** of M.

- Let M^* be the right (left) E-module $\mathrm{Hom}_E(M, E)$. The duality between projective and injective resolutions implies the following relation between the graded Bass numbers of a module and the graded Betti numbers of its dual:
  $\beta_{i,j}(M) = \mu_{i,n-j}(M^*)$, for all i, j.

Considerations
If rank $F = 1$ with $f_1 = 0$, *i.e.*, $F = E$ and $M = I$ is a graded ideal of E, then $\mathrm{Hom}_E(E/I, E) \simeq 0 : I$, where $0 : I$ is the annihilator of I. If I is a lex ideal in E, then $0 : I$ is a lex ideal in E.

Graded Bass numbers

Considerations
Let us consider the dual module $\mathrm{Hom}_E(F/L, E)$, where $L = \oplus_{t=1}^{r} I_t g_t$ is lex submodule of F. Even though the annihilators above are lex ideals, the submodule $N = \oplus_{t=1}^{r}(0 : I_t)g_t$ is not a lex submodule of F. Conversely,

$$\widetilde{N} = (0 : I_3)g_1 \oplus (0 : I_2)g_2 \oplus (0 : I_1)g_3$$

is a lex submodule in F. Note that $(F/L)^* \simeq N \simeq \widetilde{N}$ as E–graded modules.

A generalization of dual "higher" Kruskal–Katona theorem
Let M be a graded submodule of E^r, $r \geq 1$. Then

$$\mu_{i,j}(E^r/M) \leq \mu_{i,j}(E^r/M^{\mathrm{lex}}),$$

for all i, j.

Maximal Bass numbers

Example

Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = E^3$. Consider the monomial submodule of F:

$$M = (e_1e_3, e_1e_2e_4)g_1 \oplus (e_1e_2, e_2e_4, e_3e_4)g_2 \oplus (e_1e_2e_3, e_2e_3e_4)g_3.$$

We have a unique lexicographic module with the same Hilbert function of M:

$$M^{\mathrm{lex}} = (e_1e_2, e_1e_3, e_1e_4, e_2e_3)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2 \oplus (e_1e_2e_3)g_3.$$

total	3	9	23	46	80	127
0	–	–	–	–	–	–
1	–	6	19	41	74	120
2	3	3	4	5	6	7

Bass diagram for F/M

total	3	12	35	74	133	216
0	–	1	4	10	20	35
1	–	8	25	54	98	160
2	3	3	6	10	15	21

Bass diagram for F/M^{lex}

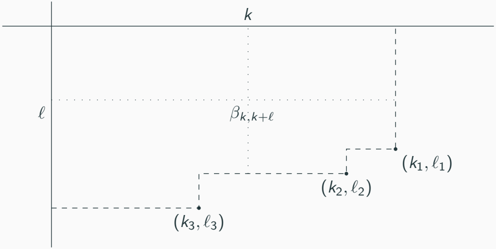
Extremal Betti numbers

Extremal Betti Numbers

- Let $S = K[x_1, \ldots, x_n]$ be the standard polynomial ring in n variables over a field K and let I be a graded ideal of S. A graded Betti number $\beta_{k,k+\ell}(I) \neq 0$ is called *extremal* if

$$\beta_{i,\,i+j}(I) = 0 \text{ for all } i \geq k,\ j \geq \ell,\ (i,j) \neq (k,\ell).$$

- The **extremal Betti numbers** (Bayer, Charalambous and Popescu [Bayer et al., 1999]) of a graded ideal I of S are the non-zero top left corners in a block of zeroes in the Macaulay diagram of I.



- They are a refinement of the *Castelnuovo-Mumford regularity* and of the *projective dimension* of the ideal I.

t–spread ideals

- Recently, Ene, Herzog, and Qureshi have introduced the notion of t–**spread monomial ideal** [Ene et al., 2019], where t is a nonnegative integer.

- Let $t \geq 0$ be an integer. A monomial $x_{i_1}x_{i_2}\cdots x_{i_d}$ with $1 \leq i_1 \leq \cdots \leq i_d \leq n$ is called t–spread, if $i_j - i_{j-1} \geq t$ for $2 \leq j \leq d$. Note that, any monomial is 0–spread, while the squarefree monomials are 1–spread.

- The ideal I is called t–**spread strongly stable**, if for all t–spread monomials $u \in I$, all $j \in \mathrm{supp}(u)$ and all $i < j$ such that $x_i(u/x_j)$ is t–spread, it follows that $x_i(u/x_j) \in I$.

- Let $u_1, \ldots, u_m$ be t–spread monomials in S. The unique t–spread strongly stable ideal containing $u_1, \ldots, u_m$ will be denoted by $B_t(u_1, \ldots, u_m)$[Ene et al., 2019]. The monomials $u_1, \ldots, u_m$ are called t– **spread Borel generators**.

Example

Let $S = K[x_1,\dots,x_8]$ and let us consider the set $P = \{x_1x_8, x_2x_6x_8\}$. We want to compute some finitely generated t-Borel ideals with the monomials in P as Borel generators.

$$B_0(x_1x_8, x_2x_6x_8) = (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2^2x_6, x_2^2x_7, x_2^2x_8, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_3x_6, x_2x_3x_7, x_2x_3x_8, x_2x_4^2, x_2x_4x_5, x_2x_4x_6, x_2x_4x_7, x_2x_4x_8, x_2x_5^2, x_2x_5x_6, x_2x_5x_7, x_2x_5x_8, x_2x_6^2, x_2x_6x_7, x_2x_6x_8);$$

$$B_1(x_1x_8, x_2x_6x_8) = (x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_2x_3x_4, x_2x_3x_5, x_2x_3x_6, x_2x_3x_7, x_2x_3x_8, x_2x_4x_5, x_2x_4x_6, x_2x_4x_7, x_2x_4x_8, x_2x_5x_6, x_2x_5x_7, x_2x_5x_8, x_2x_6x_7, x_2x_6x_8);$$

$$B_2(x_1x_8, x_2x_6x_8) = (x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_2x_4x_6, x_2x_4x_7, x_2x_4x_8, x_2x_5x_7, x_2x_5x_8, x_2x_6x_8).$$

Extremal Betti numbers

- If I is a t–spread strongly stable ideal, there exists a formula to compute the graded Betti numbers of I [Ene et al., 2019, Corollary 1.12]:

$$\beta_{k,\,k+\ell}(I) = \sum_{u \in G(I)_\ell} \binom{\max(u) - t(\ell-1) - 1}{k}.$$

- Let I be a t–spread strongly stable ideal of S.
  The following conditions are equivalent:
  (1) $\beta_{k,\,k+\ell}(I)$ is extremal;
  (2) $k + t(\ell - 1) + 1 = \max\{\max(u) : u \in G(I)_\ell\}$ and
  $\max(u) < k + t(j-1) + 1$, for all $j > \ell$ and for all $u \in G(I)_j$.

- If I is a t–spread strongly stable ideal of S and $\beta_{k,\,k+\ell}(I)$ is an extremal Betti number of I, then we have the following bound:

$$1 \le \beta_{k,k+\ell}(I) \le \binom{k+\ell-1}{\ell-1}.$$

A numerical characterization

A question:

- Given three nonnegative integers t, n, r ($n \geq 2$ and $1 \leq r \leq n-1$),
- r pairs of positive integers (k_1, ℓ_1), …, (k_r, ℓ_r) such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$, $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$,
- r positive integers $a_1, \ldots, a_r$,

under which conditions does there exist a t–spread ideal I of $S = K[x_1, \ldots, x_n]$ such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, …, $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

Some positive answers (when $\mathrm{char}(K) = 0$):

- has been given, for $t = 0$, in [Crupi and Utano, 2003], [Herzog et al., 2014] and [Amata and Crupi, 2019a],
- for $t = 1$, in [Amata and Crupi, 2019d]
- and, for $t = 2$, in [Amata and Crupi, 2019b] has been studied the maximal number of extremal Betti numbers allowed for 2–spread strongly stable ideals in S.

Case $t = 0$ [Amata and Crupi, 2019a]

Given two positive integers n, r, r pairs $(k_1, \ell_1), (k_2, \ell_2), \dots, (k_r, \ell_r)$ and r positive integers $a_1, a_2, \dots, a_r$ respecting the previous hypothesis and let K be a field of char 0, then the following conditions are equivalent:

- there exists a strongly stable ideal $I \subsetneq S$, with extremal Betti numbers $\beta_{k_i, k_i+\ell_i}(I) = a_i$, for $i = 1, \dots, r$;
- set $t = \max\{i : \ell_i \le r - i\}$. The integers a_i satisfy the conditions:

  $$1 \le a_i \le |A_i \setminus \mathrm{LexShad}^{\ell_i - \ell_{i-1}}(A_{i-1})|, \quad \text{for} \quad i = 1, \dots, r,$$

  where $A_0 = \emptyset$,

  (i) $A_1 = \{u \in A(k_1, \ell_1) : u \ge_{\mathrm{lex}} x_{k_r - 1} x_{k_1+1}\}$, whenever $\ell_1 = 2$;

  (ii) $A_i = \{u \in A(k_i, \ell_i) : u \ge_{\mathrm{lex}} x_{k_r} x_{k_r - 1} \cdots x_{k_{r-\ell_i+3}} x_{k_{r-\ell_i+2}-1} x_{k_i+1}\}$, for $i = 1, \dots, t$, whenever $\ell_1 \ge 3$, and for $i = 2, \dots, t$, whenever $\ell_1 = 2$;

  (iii) $A_i = \{u \in A(k_i, \ell_i) : u \ge_{\mathrm{lex}} x_{k_r} x_{k_r - 1} \cdots x_{k_{i+1}} x_{k_i+1}^{\ell_i - (r-i)}\}$, for $i = t+1, \dots, r-1$;

  (iv) $A_r = \{u \in A(k_r, \ell_r) : u \ge_{\mathrm{lex}} x_{k_r+1}^{\ell_r}\}$,

  and if $a_i = |[u, v]|$, with $u, v \in A_i$, then $1 \le a_{i+1}$ $\le |A_{i+1} \setminus \mathrm{LexShad}^{\ell_{i+1} - \ell_i}([u, v])|$, for all $i = 1, \dots r - 1$,

with $2 < r \le n - 2$ (it has to be $n \ge 5$), $k_r \ge 2$, whenever $\ell_1 = 2$, and $1 \le r \le n - 1$, $k_r \ge 1$, whenever $\ell_1 \ge 3$.

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```
Loading CoCoA packages from directory:
  /cocoa-5.2/packages

With CoCoALib and external libraries GMP, Normaliz, Frobby
indent(VersionInfo()); -- for information about this version


/* */ use R::=QQ[x[1..7]];
/* */ Corners:=[[5,3],[4,4],[3,5],[1,7]];
/* */ a:=[1,2,5,1];
/* */ I:=StronglyStableIdealEB(R,Corners,a,1);
Strongly Stable Ideals has been computed:
Generators:  [x[1]^3, x[1]^2*x[2], x[1]^2*x[3], x[1]^2*x[4], x[1]^2*x[5], x[1]^2*x[6],
x[1]*x[2]^3, x[1]*x[2]^2*x[3], x[1]*x[2]^2*x[4], x[1]*x[2]^2*x[5], x[1]*x[2]*x[3]^2,
x[1]*x[2]*x[3]*x[4], x[1]*x[2]*x[3]*x[5], x[1]*x[2]*x[4]^3, x[1]*x[3]^4, x[1]*x[3]^3*x[4],
x[1]*x[3]^2*x[4]^2, x[1]*x[3]*x[4]^3, x[1]*x[4]^4, x[2]^7]
Borel Generators:  [x[1]^2*x[6], x[1]*x[2]*x[3]*x[5], x[1]*x[4]^4, x[2]^7]
-1\**-  *cocoa5*    All L10    (cocoa5-comint:run)------------------------------------

-1\**-  Extremal_Betti.cocoa5    Top L1    (CoCoA5)-----------------------------------
```

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```
/* */ IsStronglyStable(I);
true
/* */ IsLexSegment(I);
false
/* */ PrintRes(I);
0 --> R[-8] --> R[-7]^6(+)R[-8]^2 --> R[-6]^15(+)R[-7]^10(+)R[-8]^5 -->
R[-5]^20(+)R[-6]^20(+)R[-7]^16 --> R[-4]^15(+)R[-5]^19(+)R[-6]^17(+)R[-8] -->
R[-3]^6(+)R[-4]^7(+)R[-5]^6(+)R[-7]
/* */ PrintBettiDiagram(I);
             0     1     2     3     4     5
- - - - - - - - - - - - - - - - - - - - - - - -
     3:      6    15    20    15     6     1
     4:      7    19    20    10     2     -
     5:      6    17    16     5     -     -
     6:      -     -     -     -     -     -
     7:      1     1     -     -     -     -
- - - - - - - - - - - - - - - - - - - - - - - -
   Tot:     20    52    56    30     8     1     /* */
```

-1**- *cocoa5*   All L10   (cocoa5-comint:run)--

-1**- Extremal_Betti.cocoa5   Top L1   (CoCoA5)--

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```
/* */ use R::=QQ[x[1..7]];
/* */ Corners:=[[5,3],[4,4],[3,5],[1,7]];
/* */ av:=AdmissibleValues(R,Corners);
/* */ av;
/* */ matrix(QQ,
[[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2],
 [1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 7, 1, 1, 2, 3],
 [1, 2, 3, 4, 5, 6, 7, 1, 2, 3, 4, 5, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 1, 1, 1, 2, 1, 1],
 [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]])
/* */
```

```
-1\**-  *cocoa5*    All L10    (cocoa5-comint:run)
```

```
-1\**-  Extremal_Betti.cocoa5    Top L1    (CoCoA5)
```

Case $t = 1$ [Amata and Crupi, 2019d]

Consider three positive integers $n \geq 5$, $\ell_1 \geq 3$ and $1 \leq r \leq n - \ell_1$, r pairs of positive integers (k_1, ℓ_1), ..., (k_r, ℓ_r) such that $n - 3 \geq k_1 > \cdots > k_r \geq 2$ and $2 \leq \ell_1 < \cdots < \ell_r$, $k_i + \ell_i \leq n$ $(i = 1, \ldots, r)$, and r positive integers $a_1, \ldots, a_r$. Let K be a field of characteristic zero. The following conditions are equivalent:

- There exists a squarefree strongly stable ideal I of $S = K[x_1, \ldots, x_n]$ with $\beta_{k_1, k_1+\ell_1}(I) = a_1$, ..., $\beta_{k_r, k_r+\ell_r}(I) = a_r$ as extremal Betti numbers;
- Setting
  (i) $v_r = x_{k_r+1} \cdots x_{k_r+\ell_r}$,
  $A_r = [w_r, v_r]$, with $w_r \in A^s(k_r, \ell_r)$ and such that $|A_r| = a_r$;
  (ii) for $i = 1, \ldots, r-1$,
  $v_{r-i} = \min\{u \in A^s(k_{r-i}, \ell_{r-i}) : \max A_{r-i+1} \notin \mathrm{BShad}(u)_{(k_{r-i+1}, \ell_{r-i+1})}\}$,
  $A_{r-i} = [w_{r-i}, v_{r-i}]$, with $w_{r-i} \in A^s(k_{r-i}, \ell_{r-i})$ and such that $|A_{r-i}| = a_{r-i}$;
  (iii) for $i = 1, \ldots, r$, $n_i = |\{u \in A^s(k_i, \ell_i) : u \geq v_i\}|$, then the integers a_i satisfy the following conditions:

  $$a_i \leq n_i.$$

  If $a_i = |[u_{i,1}, u_{i,a_i}]|$, $u_{i,j} \in A^s(k_i, \ell_i)$ $(j = 1, \ldots, a_i)$ and $p_i = |\{v \in A^s(k_i, \ell_i) : v > u_{i,1}\}|$, then $a_i \leq n_i - p_i$, for $i = 1, \ldots, r$.

A new formula

Lemma
Let n and $q \geq 1$ be two positive integers such that $n \geq q$. Then

$$\binom{n}{q} = \binom{n-1}{q-1} + \binom{n-2}{q-1} + \cdots + \binom{q-1}{q-1}.$$

Example

- Let $S = K[x_1, \ldots, x_9]$ and consider the monomial $u = x_2x_5x_7x_8$. We have that $|A^s(4,4)| = \binom{7}{3} = 35$.
- The following scheme summarizes the calculations of $|[x_1x_2x_3x_8, u]|$.

$$\binom{7}{3} = \boxed{\binom{6}{2}} + \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$$

$$\binom{5}{2} = \boxed{\binom{4}{1} + \binom{3}{1}} + \binom{2}{1} + \binom{1}{1}$$

$$\binom{2}{1} = \boxed{\binom{1}{0}} + \binom{0}{0}$$

- Then $|[x_1x_2x_3x_8, u]| = (15 + 4 + 3 + 1) + 1 = 24$.

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```
Macaulay2, version 1.14
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "SquarefreeIdeals";

i2 : S=QQ[x_1..x_10];

i3 : g={x_2x_8, x_3x_4x_5, x_3x_4x_8x_9, x_3x_5x_7x_9, x_4x_5x_6x_7x_8x_9x_10};

i4 : I=squarefreeStronglyStableIdeal ideal g
o4 = ideal (x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_2x_3, x_2x_4, x_2x_5,
           x_2x_6, x_2x_7, x_2x_8, x_3x_4x_5, x_3x_4x_6x_7, x_3x_4x_6x_8, x_3x_4x_6x_9,
           x_3x_4x_7x_8, x_3x_4x_7x_9, x_3x_4x_8x_9, x_3x_5x_6x_7, x_3x_5x_6x_8,
           x_3x_5x_6x_9, x_3x_5x_7x_8, x_3x_5x_7x_9, x_4x_5x_6x_7x_8x_9x_10)
o4 : Ideal of S

i5 : minimalBettiNumbersIdeal I
             0  1   2   3  4  5 6
o5 = total: 26 94 154 139 71 19 2
         2: 13 42  70  70 42 14 2
         3:  1  2   1   .  .  . .
         4: 11 47  80  68 29  5 .
         5:  .  .   .   .  .  . .
         6:  .  .   .   .  .  . .
         7:  1  3   3   1  .  . .
o5 : BettiTally
```


2016. Franziska Hinkelmann, Lars Kastner, Mike Stillman.

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```
i6 :  corners=extremalBettiCorners I
o6 = {(6, 2), (5, 4), (3, 7)}
o6 :  List

i7 :  r=#corners;
i8 :  a={2,5,1};

i9 :  Bg=extremalBettiMonomials(S,r,corners,a)
o9 = {x_1x_8, x_2x_8, x_3x_4x_5x_9, x_3x_4x_6x_9, x_3x_4x_7x_9, x_3x_4x_8x_9, x_3x_5x_6x_9,
      x_4x_5x_6x_7x_8x_9x_10}
o9 :  List

i10 :  J=squarefreeStronglyStableIdeal ideal Bg
o10 = ideal(x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_2x_3, x_2x_4, x_2x_5,
            x_2x_6, x_2x_7, x_2x_8, x_3x_4x_5x_6, x_3x_4x_5x_7, x_3x_4x_5x_8, x_3x_4x_5x_9,
            x_3x_4x_6x_7, x_3x_4x_6x_8, x_3x_4x_6x_9, x_3x_4x_7x_8, x_3x_4x_7x_9,
            x_3x_4x_8x_9, x_3x_5x_6x_7, x_3x_5x_6x_8, x_3x_5x_6x_9, x_4x_5x_6x_7x_8x_9x_10)
o10 :  Ideal of S

i11 :  minimalBettiNumbersIdeal J
                0   1    2    3   4   5  6
 o11 =  total: 27  97  157  140  71  19  2
            2: 13  42   70   70  42  14  2
            3:  .   .    .    .   .   .  .
            4: 13  52   84   69  29   5  .
            5:  .   .    .    .   .   .  .
            6:  .   .    .    .   .   .  .
            7:  1   3    3    1   .   .  .
o11 :  BettiTally
```

Case $t = 2$ [Amata and Crupi, 2019b]

First odd cases

n	Corner sequence	2–spread strongly stable ideal
5	$\{(2,2)\}$	$B_{2,5,1} = B_2(x_1x_5) = (x_1x_3, x_1x_4, x_1x_5)$
7	$\{(4,2),(2,3)\}$	$B_{2,7,1} = B_2(x_1x_7, x_2x_4x_7)$
9	$\{(6,2),(4,3),(2,4)\}$	$B_{2,9,1} = B_2(x_1x_9, x_2x_4x_9, x_2x_5x_7x_9)$

Theorem
Let $n \geq 11$ an odd integer and $\ell_1 = 2$. Given $\frac{n-3}{2}$ pairs of positive integers

$$(k_1, \ell_1), (k_2, \ell_2), \dots, (k_{\frac{n-3}{2}}, \ell_{\frac{n-3}{2}}), \tag{5.1}$$

with $1 \leq k_{\frac{n-3}{2}} < k_{\frac{n-3}{2}-1} < \cdots < k_1 \leq n-3$ and $2 = \ell_1 < \ell_2 < \cdots < \ell_{\frac{n-3}{2}} \leq \frac{n-1}{2}$, then there exists a 2–spread strongly stable ideal I of S of initial degree ℓ_1 and with the pairs in (5.1) as corners if and only if $k_i + 2(\ell_i - 1) + 1 = n$, for $i = 1, \dots, \frac{n-3}{2}$.

Case $t = 2$ [Amata and Crupi, 2019b]

First even cases

n	Corner sequence	2–spread strongly stable ideal
4	$\{(1,2)\}$	$B_{2,4,1} = B_2(x_1x_4) = (x_1x_3, x_1x_4)$
6	$\{(3,2),(1,3)\}$	$B_{2,6,1} = B_2(x_1x_6, x_2x_4x_6)$
8	$\{(5,2),(3,3)\}$	$B_{2,8,1} = B_2(x_1x_8, x_2x_4x_8)$
10	$\{(7,2),(5,3),(3,4)\}$	$B_{2,10,1} = B_2(x_1x_{10}, x_2x_4x_{10}, x_2x_5x_7x_{10})$

Theorem
Let $n \geq 12$ an even integer and $\ell_1 = 2$. Given $\frac{n-4}{2}$ pairs of positive integers

$$(k_1, \ell_1), (k_2, \ell_2), \dots, (k_{\frac{n-4}{2}}, \ell_{\frac{n-4}{2}}), \tag{5.2}$$

with $1 \leq k_{\frac{n-4}{2}} < k_{\frac{n-4}{2}-1} < \cdots < k_1 \leq n-3$ and $2 = \ell_1 < \ell_2 < \cdots < \ell_{\frac{n-4}{2}} \leq \frac{n-2}{2}$, then there exists a 2–spread strongly stable ideal I of S of initial degree ℓ_1 and with the pairs in (5.2) as corners if and only if $k_i + 2(\ell_i - 1) + 1 = n$, for $i = 1, \dots, \frac{n-4}{2}$.

RESET INTERRUPT SAVE UPLOAD FILE

```
i1 : loadPackage "SquarefreeIdeals";
i2 : n=13;
i3 : S=QQ[x_1..x_n];
i4 : t=2;
i5 : indeg=2;
i6 : k=n-t*(indeg-1)-1
o6 : 10
i7 : tot=(k-k%t)//t
o7 : 5
i8 : corners=for i to tot-1 list (k-t*i,indeg+i)
o8 : {(10, 2), (8, 3), (6, 4), (4, 5), (2, 6)}
o8 : List
i9 : a=toList(#corners:1);
i10 : Bg=tspreadExtremalBettiMonomials(S,corners,a,t)
o10 = {x_1x_13, x_2x_4x_13, x_2x_5x_7x_13, x_2x_5x_8x_10x_13, x_3x_5x_7x_9x_11x_13}
o10 : List
```

RESET INTERRUPT SAVE UPLOAD FILE

```
i14 :  I=tspreadStronglyStableIdeal(t,ideal Bg)

o14 = ideal (x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_1x_9, x_1x_10, x_1x_11,
           x_1x_12, x_1x_13, x_2x_4x_6, x_2x_4x_7, x_2x_4x_8, x_2x_4x_9, x_2x_4x_10,
           x_2x_4x_11, x_2x_4x_12, x_2x_4x_13, x_2x_5x_7x_9, x_2x_5x_7x_10,
           x_2x_5x_7x_11, x_2x_5x_7x_12, x_2x_5x_7x_13, x_2x_5x_8x_10x_12,
           x_2x_5x_8x_10x_13, x_3x_5x_7x_9x_11x_13)

o14 :  Ideal of S

i15 :  minimalBettiNumbersIdeal I

              0   1   2   3   4   5   6   7  8  9 10
 o15 = total: 27 120 294 496 610 553 367 174 56 11  1
           2: 11  55 165 330 462 462 330 165 55 11  1
           3:  8  36  84 126 126  84  36   9  1  .  .
           4:  5  20  35  35  21   7   1   .  .  .  .
           5:  2   7   9   5   1   .   .   .  .  .  .
           6:  1   2   1   .   .   .   .   .  .  .  .

o15 :  BettiTally
```

Future works

Future works

- We intend to implement improvements to the *Macaulay2* package **ExteriorModules**. More precisely, given a submodule of F we would like to implement some algorithms
  - to compute the **Generic Initial Module**
  - to manage the **Dual Module** (in a general case)
- Given an integer $t \geq 2$, let $\mathcal{S}_{t,n}$ be the set of all t–**spread strongly stable ideals** in S. What is the largest number of corners allowed for an ideal of $\mathcal{S}_{t,n}$?
- Given three positive integers $t \geq 2$, n and $r < n$, r pairs of positive integers (k_1, ℓ_1), …, (k_r, ℓ_r) such that $n - 3 \geq k_1 > \cdots > k_r \geq 2$ and $2 \leq \ell_1 < \cdots < \ell_r$, and r positive integers $a_1, \ldots, a_r$, under which conditions does there exist a t–spread strongly stable ideal I of $S = K[x_1, \ldots, x_n]$ such that $\beta_{k_1, k_1+\ell_1}(I) = a_1$, …, $\beta_{k_r, k_r+\ell_r}(I) = a_r$ are its **extremal Betti numbers**?

That’s all Folks... for now.

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